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THE ART OF SNARING DRAGONS

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ABSTRACT

DRAGONS are formidable problems in elementary mechanics not amenable to solution by naive formula cranking. What is the intellectual weaponry one needs to snare a Dragon? To snare a Dragon one brings to mind an <u>heuristic</u> frame – a specifically structured association of problem solving ideas. Data on the anatomy of <u>heuristic</u> frames – just how and what ideas are linked together – has been obtained from the protocols of many attacks on Dragons by students and physicists. In this paper various <u>heuristic</u> frames are delineated by detailing how they motivate attacks on two particular Dragons, *Milko* and *Jugglo*, from the writer s compilation. This model of the evolution of problem solving skills has also been applied to the interpretation of the intellectual growth of children, and in an Appendix we use it to give a cogent interpretation for the protocols of Piagetian "Conservation" experiments. The model provides a sorely needed theoretical framework to discuss teaching strategies calculated to promote problem solving skills.

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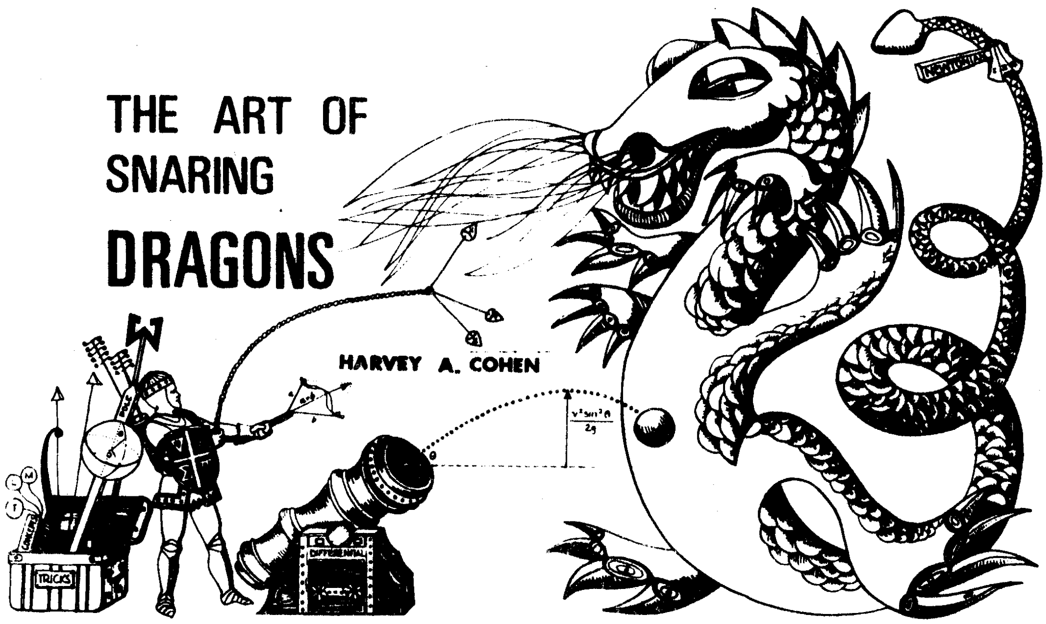

0. INTRODUCTION

Physicists tackle problems. Students also tackle problems. But in what way does the tyros efforts differ from the experts? This paper grew out experiences in teaching a college-level course in mechanics, when I found that students might "follow" line by line demonstrations examples, yet had little belief or understanding in the qualitative conclusions. I probed deeper, and found, that there were many problems which could be solved qualitatively, with answers such as "The same", "More", "Less" which with some coaxing, or even counterargument, by the presenter, students could reach a confident conclusion. Yet this confident, "naive" solution of the tyro was clearly wrong to an "enlightened expert". I use the term "enlightened expert" advisedly, as I found that I could devise problems that even professional physicists, would, on initial encounter, give the same answer as tyro physicists. But the experts, when advised of their error, would (mostly) rapidly come to see an inadequacy in their initial effort, and develop the (correct) expert response.

This preliminary experience lead me to devise a cluster of problems, in a small componendium10, *What G Killed Ned Kelly*? The problems were illustrated, to stimulate the imagination, and were augmented by hints, suggestions, and counter-suggestions, introduced by a bevy of whimsical characters aligned with the ""Dragon" theme. The title was based on a question I had devised seeking a mechanical analysis of an actual historical event: the outrageous story of how the ("last") bushranger Ned Kelly was hung at the Old Melbourne Goal in 1876 -- not just once, but four times ain all, using ropes of various length by an executioner of an amazing experimental character. But most of the problems I devised were less macabre and challenging in quite a different sense: essentially qualitative, requiring no more than answer such as "The Same" or "More/Less" yet such that provided with a modest amount of discussion, students could confidently give an answer. And so could my academic colleagues. But mostly the students, and surprisingly often some of my colleagues, confidently asserted the wrong answer. What was going on here?

It seemed that when faced with familiar problems, for which the outline of a solution is familiar, traditional experts can regurgitate what they have learnt. However when faced with a problem in mechanics that is unfamiliar, both physicists as well as tyros tend to have recourse to general problem solving schemes (later in this paper termed <u>heuristics</u> frames, or simply <u>heuristics</u>, which may require elaboration, amendment to achieve a proper solution.

I turned for inspiration to recent work in the history and philosophy of science. The noted philosopher of the history of mathematics, Imre Lakatos, pointed out [1] that the historical evolution of a mathematical proof was furthered by the invention/discovery of what Lakatos termed "Monsters," counter-examples with bizarre feature not conceived of by the originators of a proof. Lakatos saw two major ways of dealing with "Monsters"

(i) Monster barring – wherethe gamut of a theorem was changed so as to exclude particular Monsters.

(ii) Monster adjustment - where the description of the Monster was altered so that it could be recognised as consistent with the (broader) intention of the proof.

The concept of Monster-barring can be readily illustrated by one of my Dragons, called "Inducia Capillaria". This Dragon is indeed one ofLakatos's Monsters, because it is a counter-example to the result that can be expressed qualitatively as:

> *The narrower a capillary, the greater the height that a column of water will be drawn up a vertical capillary.*

(See below).Monster barring can be applied by rephrasing the result as

> *The narrower a capillary, the greater the height that a column of water will be drawn up a vertical capillary, provided that the column height above external water level does not exceed the vertical height of the capillary (above the same datum).*

so that the derivation needs to be rather more careful to lead to this outcome.

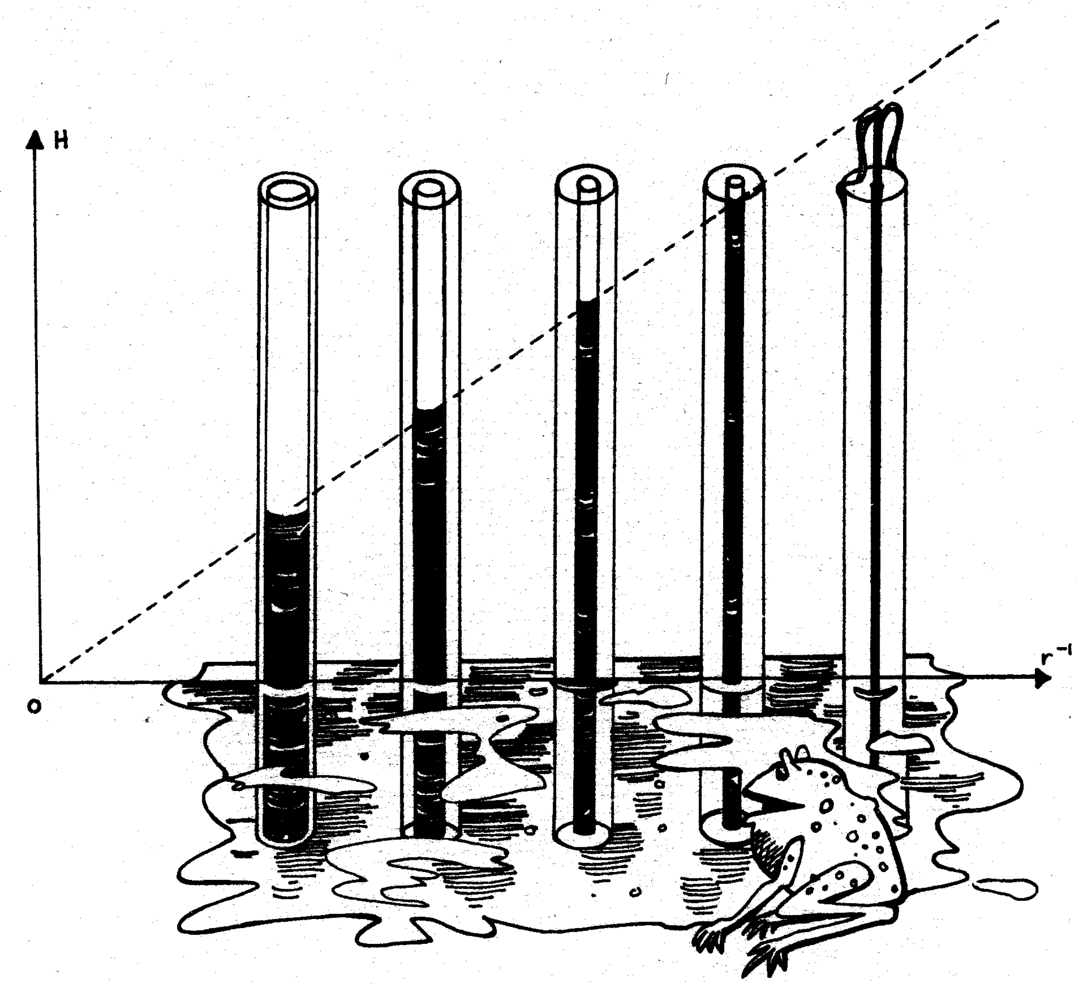

Fig: "Inducia Capillaria – 1". This diagram serves is aLakatosian Monster to a derivation presented in many elementary texts: that the height H of a column of liquid rises (or falls) up (or down) a (vertical) capillary tube is inversely proportional to the inner radius of the capillary. Formally H = K/r . This Monster is reproduced from H.A. Cohen, *A Dragon Hunter's Box*, Hanging Lake Books, Warrandyte, Victoria 1974 with Permission.

I found that monster barring is a problem solving strategy much used by students of physics, who use it to carry simultaneously a growing knowledge of physics, whilst at the same time holding notions that are contrary to it. An fine example was provided by responses to demonstrations of procession.

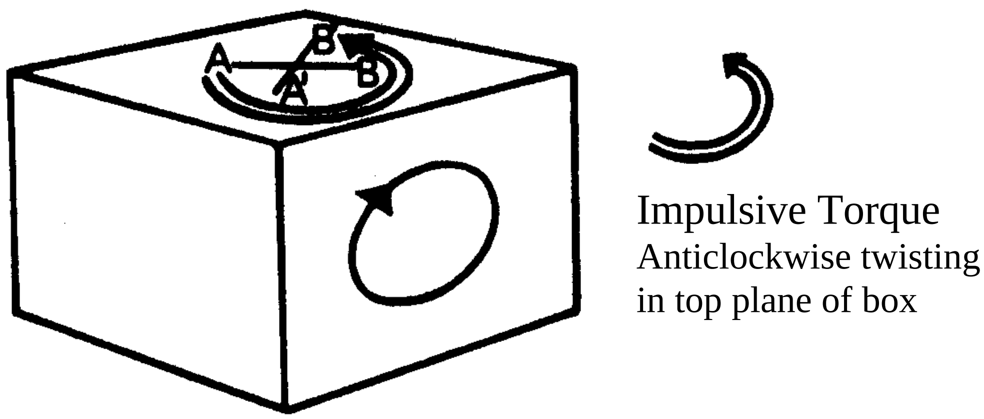

Fig (i) Student Expectation re gyroscopic effect. A flywheel is spinning in the bax in a vertical plane. The projection of the flywheel in the top plane. Following an impulsive torque, a twisting about the vertical, students (falsely) expect this projection to rotate, as to A'B'

Students say that such precessional behaviour is counter-intuitive, and apply Monster barring to separate their common-sense thinking from this issue. But it is clear that a student who disbars such a problem area by monster-barring must perform more radical mental gymnastics to become an effective problem solver within such an area.

So what can the philosophers of science say about radical reorganization of knowledge? Lakatos further developed his pioneering ideas on the evolution of mathematical "proofs" to encompass broader realms of science in terms of "researchprogrammes"[2]. But the history of mechanics has been most insightfully developed by Thomas Kuhn, in *The Copernican Revolution*[3]. In *The Structure of Scientific Revolutions*[4], Kuhn insisted that the history of science is based on periods of "normal science" in-between which there are periods of "revolutionary" ferment. It is during "normal science", that scientists steadily extend the range and application of scientific theories. During such a period, there are some puzzles that seem hard to solve or understand, but there is confidence that with a little more work these quirky results will be explained. But each normal science period ends with a revolution, after which the quirky puzzles are seen as counter-examples to the previous established body of theory. During the revolution, a often bitter dispute rages between the advocates of the old and the new.

It became clear to me that for the individual student of physics, some of the Dragons, were just the quirky problems whose solution required a radical reorganisation of a students problem solving skills. It further appeared that problems such as the Dragon *Milko* discussed in Section 2, were strongly analogous to Paigetian conservation puzzles. Seymour Papert has provided a cogent exposition of certain 'classic' Paigetian Conservation experiments, that probe the intellectual development of young children. Largely inspired by Papert, and with inspiration from Minsky's Frames concept, a formalism has been developed to describe the cluster of problem solving ideas as a <u>heuristic,</u> where the underline nature of the term is meant to imply a multi-dimensional quantity, as is similarly done to denote three-dimensional vectors.

In Table I the components of a <u>HEURISTIC</u> are delineated.

TABLE !	
THE ANATOMY OF A <u>HEURISTIC</u>	
COMPONENT	DESCRIPTION
(Core) heuristic	An elemental, crude problem solving idea, probably acquired in childhood.
Problem Reduction Devices and Algorithm Selector	How to reshape the problem and which algorithm to apply.
Debug routines	What to do when things "go wrong".
Demons: Warnings, Caveats, Flags, Pointers	Miscellaneous: "Watch out". Limitations. "Try another <u>heuristic</u> frame"

In Table 1 and elsewhere in this paper, by an algorithm is meant a highly specific procedure or formula. The (core) heuristic of a <u>heuristic</u> is the same sort of mental object as what Polya[7] termed a heuristic – a problem solving idea of some potency. (Polya confined his attention to mathematics, however). Problem reduction involves putting the problem in a form suitable for the application of particular algorithms. If the unexpected happens – or even when one is informed that the answer derived is "wrong" – one calls upon the Debug Routines of the <u>heuristic</u>. Also linked with the other components of a <u>heuristic</u> are what I ve termed Demons: the image is of some little beast that waits for some specific little occurrence to trigger his attention – when he passes on his message. At any rate, under the heading of "Demons" are lumped together some miscellaneous ideas bound in the <u>heuristic</u>, such as warnings, caveats, and directives to other frames. A few examples of Demons are presented later in this paper.

1. A TEACHING STRATAGEM

Every teacher of physics would, I feel, assert that he strives to teach students not only how to solve certain paradigm examples – but that he also hopes to impart a cluster ofgeneralized skills in problem

Every teacher of physics would, I feel, assert that he strives to teach students not only how to solve certain paradigm examples – but that he also hopes to impart a cluster ofgeneralized skills in problem solving that will equip students to comprehend and analyse a greater range of problems than could possibly be discussed in lecture classes and tutorials. But is there any specific way to achieve this objective? The purpose of this introduction is to recount some of the more accessible ideas in a teaching stratagem I have been developing for this purpose. This stratagem shares elements in common with what I call the Fermi stratagem (in Physics teaching) and the Polya stratagem (in Mathematics teaching).

In the Fermi stratagem students are posed problems of a more project-like character. Some such Fermi problems are relatively open-ended e.g., "How, in terms of Physics, do we walk and run?". Other Fermi problems have a definite solution but are of a "non-standard" form requiring the skilful selection and artful selection of perhaps quite elementary physical models. Teachers wishing to follow the Fermi stratagem face two difficulties. The first is the scarcity of Fermi problems, or rather the scarcity of compilations of such problems. In this regard, Walker s "Flying Circus"[5] is a very welcome addition to the Physics teaching literature. This writer has also compiled a collection of non-standard problems (which he calls "Dragons") in elementary mechanics, *A Dragon Hunter s Box* of which some of the Dragons may be aptly characterised as Fermi problems[6]. The second difficulty in introducing Fermi problems is the absence of any comprehensive tutor s guide embodying theoretical analysis and practical experience in the presentation and effective utilization of such problems. In fact the casual introduction of Fermi problems into certain innovatory courses of recent years has often lead to obvious failure, as the students participating have lacked any model of how to proceed in tackling any problem other than those more conventional problems which I call "formula crankers".

In the Polya stratagem, students already familiar with the tricks and techniques needed for particular problems, are given specific instruction in powerful general problem solving ideas - - what Polya terms mathematical heuristics. The style of presentation, as evidenced by the structure of *Mathematics and Plausible Reasoning*[7] is to first explain a particular heuristic, and demonstrate its applicability to a particular problem: the student is then posed a graded set of problems which are amenable to solution via that heuristic. The writer is not aware of any extensive application of the Polya stratagem to Physics teaching.

My own teaching stratagem grew out of an attempt to implement the key ideas of Fermi and Polya in the context of a college course in elementary mechanics. I was especially keen to get away from the traditional emphasis on problems which may be characterised as "formula-crankers" and to engage students in problems which had more of the flavour of research problems[8] in Physics, such as Fermi problems. There are in fact very few published Fermi problems in elementary mechanics, and it seems that the typical problem actually posed by Fermi was "How many piano tuners are there in New York"[9] So in order to produce a significant compilation of challenging problems for student use I was obliged to devise a number of new problems in elementary mechanics which I termed Dragons to express their formidable character. In line with this playful terminology, the first compilation of Dragons was produced in a hand lettered and illustrated booklet [10] entitled "What G killedNed Kelly ? and Other Problematical Dragons" (Ned Kelly – an Australian folk hero – the last of the bushrangers – was hung in Melbourne in 1867). The "Ned Kelly" Dragon book was used in conjunction with the lectures and tutorials of a course in elementary mechanics.[11]

It is now opportune to discuss the pedagogic principles underlying the selection and construction of the Dragons of the original compilation [10]and its successor[6].. The Dragons were conceived as providing scope for the discussion of problem solving per se rather than particular physical principles. An underlying assumption was that many students try to solve problems in accord with the following model:

<u>The Formula Cranker s Model</u>

Step 1.   Look at P, the problem to be solved
Step 2.   Scan one s repertoire of all the problems one can solve, until one finds S similar to P.
Step 3.   Apply the algorithm used to solve S, to solve P.

I ve called this model the Formula Cranker s Model of Problem solving as this model will, in fact, be of some real service to a student in the solution of a formula cranker - - a problem in which has been specified precisely those elements to be substituted in a familiar formula: for instance if shape parameters (such as might be involved in a moment of inertia) are not explicitly labelled and specified the "similar" problem must not devolve on such parameters. My Dragons were selected or constructed so that like the real problems tackled in research the Formula Cranker s Model would fail. Consider first the Dragon *Milko* of Fig (iv). Because *Milko* explicitly seeks the determination of the pressure at the bottom of a cylinder-like volume (the interior of a milk bottle), this Dragon is clearly "similar" to the calculation of the pressure at the base of a cylindrical column of liquid. In this sense the Dragon is also "similar" to other calculations of base pressure such as the pressure upon the sole of ones shoes. Hence applying to *Milko* the algorithm of the "similar" problems, the base pressure P is given in terms of the base area A and the total weight of the contents of the bottle, W, as

$$p = W/A$$

This expression is entirely false, and is an instance of how the Formula Cranking Model can lead to an inappropriate formula. Consider next the Dragon *Jugglo* of Fig (xiv). It happens that this Dragon may be successfully snared using the same formulas as are applied to the calculations of the mechanics of a rigid body. Yet as juggling is in no sense "similar" to a rigid body, students following the Formula Cranker s Model of action will not arrive at such an analysis (as is given under the caption "In Toto" in Section 3.) That is, by this example, we see how the Formula Cranker s Model may prevent students from recognising the applicability of quite familiar algorithms. The third point to be made about the Formula Cranker s Model is that even if following this model one determined an appropriate algorithm, application to the given problem may lead to a mess of algebra which is hard to untangle to finally solve the problem. An illustration of this sort of phenomena is provided in Section 2, below the heading Formula Crank.

The above examples indicate that exposure to those formidable (yet elementary) problems I ve termed Dragons highlights to students the inadequacy of the Formula Cranker s Model of Problem Solving. But in fact this is only a minor aspect of what can be learnt from such encounters. Particularly when one has in fact produced the canonical wrong answer to a Dragon, a study of such encounters, using introspection and observation of other students, reveals the sort of mental construct – collection of associated ideas – one has brought to bear on the problem.

How in fact does one solve physics problems ? Over the past few years I ve listened intently to many attempts by students and physicists to snare the Dragons of my collection[6] These observations (protocols is the jargon word in psychology) support the contention that in solving such problems one uses a structured collection of associated ideas that I ve termed a <u>heuristic</u> frame. There appears to be only a relatively small number of <u>heuristic</u> frames available to any individual, of the order of twenty. In Table 1, the anatomy of a <u>heuristic</u> frame is revealed..

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The concept of Heuristic frames provides a description of the evolution of problem solving skills in terms of

a) The growth of one s repertoire of algorithms.
b) The elaboration and augmentation of the components of one s heuristic frames.

The latter process is termed the ‘debugging of heuristics : in debugging the core heuristic is essentially unalterable, only the other components of the frame can be edited. A simple description of problem solving in terms of the components of heuristic frames is contained in a model which is called the Horse and Cart or HAC Model (of problem solving).

TABLE II	
HORSE AND CART MODEL OF PROBLEM SOLVING (H.A.C.)	
TO HAC :	
Step 1.	Given a problem, choose a **Heuristic**
Step 2	Reformulate the problem and select an **A**lgorithm.
Step 3.	**C**rank the algorithm.
Step 4.	In case of trouble, DEBUG.

The HAC model is presented in Table II. This model essentially states that the choice of Heuristic precedes the choice of an Algorithm that does the actual Cranking of a problem. As stated above, the model is over simple, but has proved to be an effective tool in promoting problem solving skill, by providing a descriptive basis for self-assessment and student counselling. This in total, this paper deals with a teaching stratagem based on two models:

i) A model for intellectual development in terms of the debugging of heuristics.
ii) A model for problem solving.

An example of how a tutor may aid the intellectual development of a student by directing attention to the debugging of one particular heuristic is provided by the following example taken from my tutorial records.

A student complained that he didn t "understand" gyroscopic effects. What was meant was that he could follow the mathematical presentation given in class, yet the behaviour was still surprising. I probed further and found that if a flywheel was spinning in a vertical plane, and a torque about the vertical axis was applied for an instant, this student expected the fly-wheel to remain vertical, but for its plane to rotate about the vertical axis.

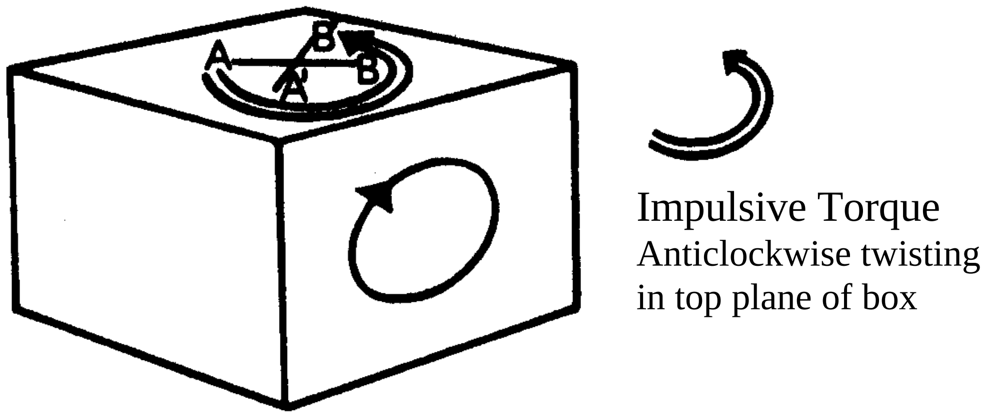


Fig (i) A (False) Student Expectation re gyroscopic effect. See Text

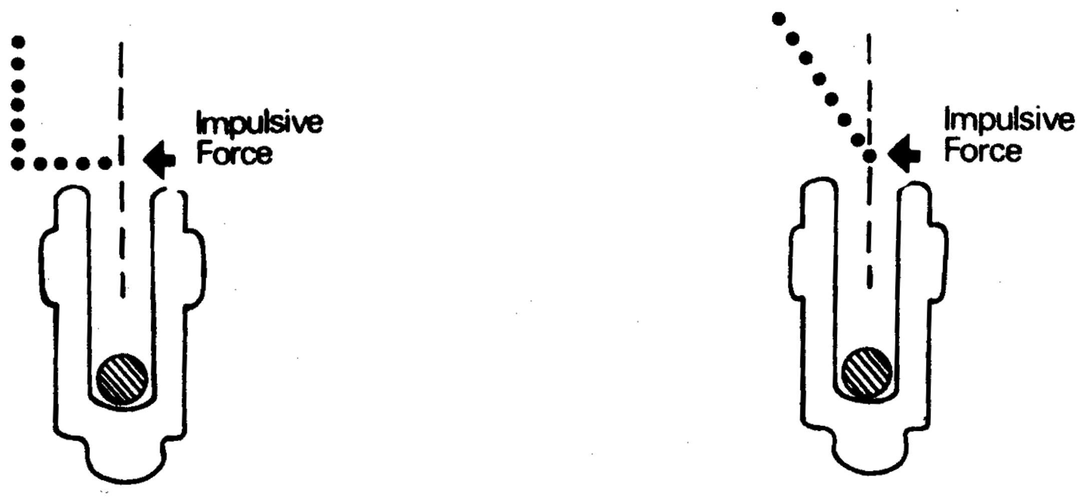
(See Fig(i)). Imagine that a spinning flywheel is placed inside the box (drawn here in isometric projection) with the plane of the flywheel parallel to the front face of the box. The spin sense of the flywheel is marked on the front face, and the projection of the wheel, the line AB, on the top of the box. A torque, applied briefly, is indicated by its tendency to twist in the top (horizontal) plane, plane, rather than as a vertical vector. One common student expectation is that the new position of the flywheel has the projection A B on the top of the box, corresponding to a rotation of the plane of the flywheel about the vertical .

Figure (i) is the diagram that was drawn while endeavouring to clarifying the student s expectation. It is clear that the student was here invoking the heuristic Parallel – the idea that the effect of a force is a displacement in the direction of that force. (The direction in this specific case is a screw sense). The student had selected an algorithm that could be formally stated as

Twisting Force x Time = Amount of Twist

This particular algorithm is appropriate to a high friction environment such as the domestic arena of a young child. It is essentially an Aristotlean algorithm – part of a physics where forces “cause” displacements in position. In order to help this student debug I constructed an argument involving the same heuristic (parallel) and patently presenting a choice between Newtonian and Aristotlean algorithms for forces:

Consider a canon firing at a target (drawn schematically from above) in Fig (ii)



A ARISTOTLEAN

B NEWTONIAN

Fig (ii) Dashed line is the unperturbed trajectory of a cannon ball. Dotted line denotes new trajectory after application of an impulsive force according to (A) Aristotlean Algorithm, (B) Newtonian Algorithm.

Suppose just as the cannon ball emerges from the barrel it is given a short sharp knock. Then, in accordance with the expectation in Fig (i) of generalised impulsive forces causing a spatial displacement in the direction of application the ball should be deviated as shown in Fig (ii)A . Now of course what actually would take place is properly demonstrated in Fig (ii) B – the effect of the impulsive force is to give the ball a transverse component of momentum to determine the subsequent trajectory of the ball. Returning to the flywheel problem, it likewise follows in formal terms that the effect of an impulsive torque about the vertical is to produce angular momentum about the vertical, which has to be compounded with the pre-existing.

This is well made by a drawing such as Fig (iii).

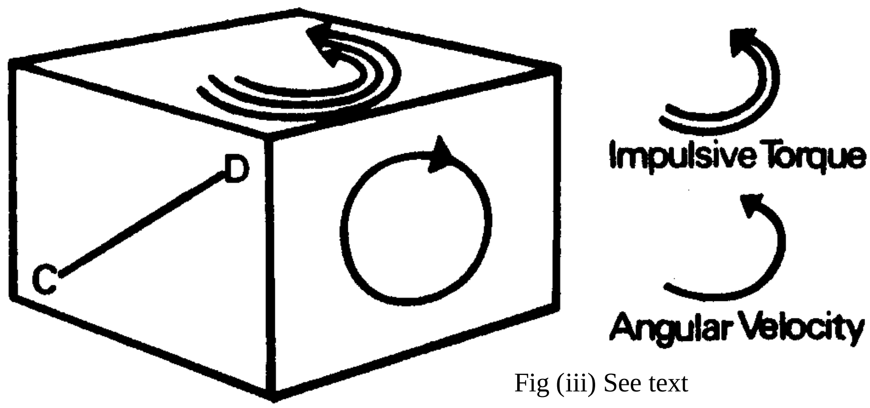


Fig (iii) See text

In this figure the original and the additional (angular) momenta are drawn as screw senses on the sides of a box containing the flywheel. But these two screw senses – compounded in a Newtonian way (algorithm) – must be just the projection of the resultant motion of the flywheel. So – imaging arrows drawn on the flywheel showing rotation sense – one deduces that the flywheel – having suffered the impulsive torque (double arrows in the figure) – changed its plane of motion: the new projection of the flywheel is shown in Fig (iii) as the straight line CD.

In summary, my first concern as a tutor was to aid this student in debugging the heuristic (Parallel) he had sought to invoke for the processional problem. (Compare computer programming: one has to debug the programs one actually writes; on the other hand it pays to learn of other programs). Confronted with this student a tutor espousing a different strategy might have replied: "Don t look at a wheel like that. Look at a wheel as composed of little parts[12], and consider the effect of the applied forces on each little part . . . ."This particular approach invokes the heuristic "Divide and Conquer" (discussed later in this paper) and it is well for a student to see a "Divide and Conquer" approach to a tantalising problem: however, to repeat, in line with the above described model for problem solving, attention to the debugging of a heuristic is paramount, and should be a tutor s first concern.

Physics problems depend on a small number of heuristics specific to physics. In this paper we are to discuss just seven of these heuristics:

- Formula Crank
- To Paradigm
- In Toto
- Fibre/Capillary
- Add Effects (and Subtract Effects)
- Divide and Conquer
- Process

In this list "Formula Crank" is none other than to apply the Formula Crankers Model of problem solving, the other heuristics are described in Section 2. For the moment it is important to note just how few there are, and that in my teaching stratagem explicit names are given to each heuristic. Now in the

Polya stratagem students gain “familiarity” with a particular heuristic by applying that heuristic to a range of different problems. In my stratagem this is also done, but much stress is also laid in applying different heuristics to the same problem – to stimulate the debugging of theseheuristics. And also to overcome what I call Magic Key Thinking **- -** the idea that there is just one way of looking at a given problem (a unique heuristic).

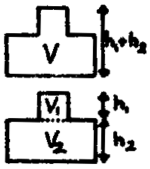
Just what are these heuristics, and how good are they in practice? Section 2 is devoted to delineating these six heuristics, and showing their application to the snaring of the Dragon *Milko* of Fig (iv). Section 3 shows how four of these heuristics motivate algorithms that successfully snare the Dragon *Jugglo* of Fig (xvi). The discussion of Section 2 and 3 will prove of value to any teacher who wishes to discuss the two Dragons, *Milko* and *Jugglo* with students – using the tutorials as heuristic debugging scenes where the tutor is equipped to guide an ill-formed but not heuristically misguided foray at these Dragons. In Section 4 the teaching stratagem presented here is reviewed. The Appendix shows the application of the theoretical framework of this paper to aspects of the intellectual development of children. The “debugging of aheuristic“ is thereby demonstrated in a simple setting, various heuristic morals are drawn.

★A milk bottle is allowed to stand so that the cream rises to the top: this occurs without any change in total volume. Does the pressure near the base of the bottle change?

★Surely, since the total weight W of the liquid remains constant, the pressure on the base, of area A, being W/A remains constant.

★After separation, the pressure immediately below the cream is less than what it was at that level before separation. So maybe...

★A mathematician uses a regular milk bottle of volume V, containing liquid of density P. After settling time, the liquid separates into two components of density P_1 and P_2, which occupy volume V_1 and V_2 respectively, as indicated, where $V_1 = h_1 A_1$, $V_2 = h_2 A_2$. So he calculates a change in pressure after separation.



★A would-be physicist challenges the mathematician: "There are no different bulk forces acting after separation; the pressure is unchanged."

★The mathematician deduces: the w.b.p. takes his milk in cartons.

Fig (iv) The Dragon *Milko*.

2. MILKO

Preliminary Remarks

The problematical Dragon *Milko* of Fig (iv) is reproduced from my compilation *A Dragon Hunter s Box.* Please read the first paragraph of this Dragon. I have posed this problem to many undergraduates, graduates, engineers and professional physicists. Invariably they jumped to the conclusion p = p ,. When informed that this was the canonical wrong answer, a line of argument often developed which made plain the heuristics invoked, and the debug routines, caveats and warnings that were associated with particular heuristics, The later paragraphs of this Dragon consist of a series of suggestions and counter-suggestions designed to provoke such an analysis by the reader. So . . . . what heuristics are there for snaring Milko, and just how is it done ?

In Toto

The heuristic "In Toto" embodies treating the diverse parts of a physical system as a single system. In the text of *Milko* the statement of the "would-be physicist" suggests that the w.b.p. - - like many first exposed to this Dragon - - has adopted an "In Toto" viewpoint and applied an elementary statics algorithm to equate the total gravitational force W to the product of base area A and base pressure.

On being informed that they have given the canonical wrong answer for *Milko*, "In Toto" champions - - who have treated the milk as a whole - - tend to

a) Check whether they have included too much in the whole
b) Check whether they have included too little in the whole
c) Switch to "Divide and Conquer" viewpoint

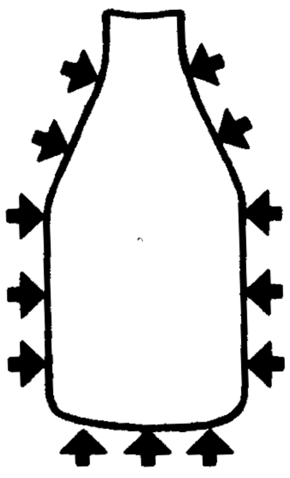


Fig (v) Sketch of wall reaction (due to pressure) forces acting on the contents of a milk bottle

The routines (a) and (b) are debug routines (or part of debug routines) associated with the "In Toto" heuristic. (c) is what I d simply call a flag or pointer to an alternative heuristic. Of course the more skilful problem solvers are more effective in invoking the above (and other) debug routines.

Debug routine (a) suggests to check what was included in the quantity W: and clearly it was the weight of the bottle, so that W/A is the pressure at the base of the bottle at the glass/table boundary. At this stage there s a strong inducement to switch to "Divide and Conquer" and check whether the pressures above and below the glass base of the bottle are equal or not. (See the discussion under the heading "Divide and Conquer")

Debug routine (b) inspires the question "Is the milk really just sitting there with just the force of gravity and base pressure (times base area) holding it in place?" This leads to the more particular question as to whether the side-wall pressure forces can have a net vertical sum. Now side-wall pressure forces don t cancel - - although they do where the walls are vertical – but not where the bottle walls are slanting. As illustrated in Fig (v) the reaction forces have a net downward sum X when the contents are homogenous, X after separation of cream.

Applying the usual statics algorithm,

Force on base before separation p A = W + X
Force on base after separation p A = W + X

When the milk separates, the density of liquid in the neck is less, so that pressures in this region are less, so that the sum of all the sidewall reaction force is less after separation:

X < X

and hence the conclusion p < p . The base level pressure drops after separation.

Divide and Conquer

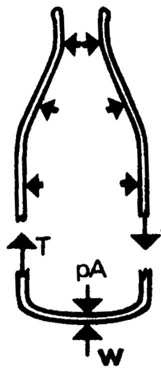


Fig vi Milk bottle broken into two sections

The heuristic that I ve called "Divide and Conquer" exhorts one to divide a physical system into a number of parts, and to solve the various sub-problems before assembling the component parts and the corresponding sub-problems. The example of this heuristic applied to snare *Jugglo* is rather more cogent than what we do here.

We take as starting point the calculation of pressure *below* the base of the bottle presented above. Break the bottle into the parts shown in Fig (vi). The vertical tension in the sidewalls of the bottle is easily overlooked. By considering the equilibrium
of the base one deduces:

W = pA - T ; W = p A - T

where T, T are the (corresponding) vertical sums of the sidewall forces at the base. By considering the equilibrium of the sides of the bottle, one deduces that T and T are exactly cancelled by the vertical sum of the forces due to liquid pressure acting on the sides, i.e.,

T = X T = X

where X(X) is the same quantity as determined in the "In Toto" discussion. Thence, on comparing X and X , one deduces that base pressure is less after separation, p < p.

Formula Crank

The heuristic "Formula Crank" involves the application of what in the introduction was called the Formula Cranker' Model of Problem Solving. To illustrate the potency of Formula Crank -–I will repeat an apocryphal story about Feynman and his early work. [13]. It appears that in a discussion Jauch informed Feynman of the 1931 paper of Dirac which showed that there was an analogy between unitary transformations in quantum mechanics and the exponential of S where S was a classical quantity. Whereupon there and then Feynman proceeded to manipulate the "analogous" classical expressions as though they were the quantum mechanical unitary transformations, to yield a first crude version of what was to become his important "Space Time Formulation of Quantum Mechanics". Clearly this was Formula Crank motivated work – but Feynman had to call upon all his intellectual resources – his elaborated (debugged) heuristics - - to make a mess of meaningless formulae into an important element of modern physics. To illustrate the impotency of Formula Crank by itself – here is how it might be applied to *Milko*. First, to recapitulate th discussion of the Introduction. A devotee of Formula Crank will take recourse to other calculations of base pressure, as of the pressure at the base of one s shoe, to calculate a constant base pressure

p = W/A

in terms of the weight, W, of the contents of the milk bottle, and base area A. If the validity of this result were queried, what would a Formula Cranker do? Very little, observation suggests. The weakness in Formula Crank is that there is no means to debug a solution other than relatively capriciously selecting a new algorithm. So as a next step, consider the application of what might be billed as the most comprehensive algorithm for calculating pressures, the formula

$$p = \Sigma_i (\rho_i h_i)g$$

where the summation is over layers of length h_i of material of density ρ_i . We apply this formula to the simplified shape “mathematical milk bottle” of Fig (vi). where subscript 1 refers to the top of the bottle, and subscript denotes the lower volume. For homogenous milk the base pressure is

$$p = (\rho_1 h_1 + \rho_2 h_2)g$$

where ρ_1 is density of cream. But this algorithm isn t enough. Conservation of matter requires that

$$\rho_1 V_1 + \rho_2 V_2 = \rho(V_1 + V_2)$$

where ρ is the density of the milk before separation. By geometry, these volumes are given in terms of areas and heights by

$$V_1 = A_1 h_1 \qquad V_2 = A_2 h_2$$

When one could deduce the clumsy formula:

$$\frac{p}{p} = \frac{(\rho_1 h_1 + \rho_2 h_2)(V_1 + V_2)}{(\rho_1 V_1 + \rho_2 V_2)(h_1 + h_2)}$$

From this formula it is clear that $p \neq p$, but it takes a measure of careful algebraic manipulation before the barest qualitative features emerge. In contrast, consider an “In Toto “ motivated attack. See Fig (vii), in which the arrows indicate the vertical forces acting on the contents of the “regular” milk bottle of the mathematician.

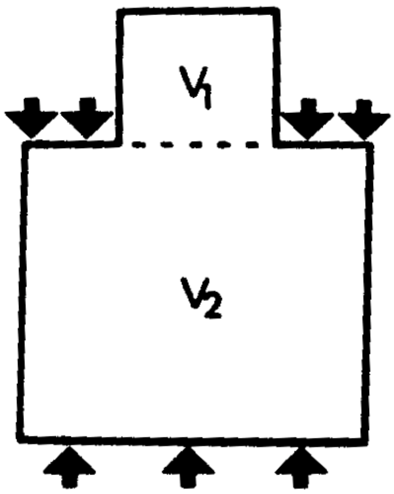


Fig (vii) Vertical wall forces acting on the contents of the “mathematicians milk bottle”.

For homogeneous milk

$$pA_2 = (V_1 + V_2)\rho g + (A_1 - A_2)h_1\rho g$$
$$= W + (A_1 - A_2)h_1\rho g$$

For stratified milk, cream in volume V_{1}, cream in volume V_{2}

$$p \; A_2 = W + (A_1 - A_2)h_1\rho_1 g$$

In this case, as cream is lighter than milk, i.e., $\rho_1 < \rho$, it follows that $p < p$. The point being made is that in an argument motivated by the heuristic "In Toto" the algorithm gets marshalled - - is interpretable and therefore under control. A Formula Cranker needs mathematical skills of high order to organize an elementary physical calculation.

Columns (Reduction Device A)

The heuristic "Fibre" is a valuable problem solving idea utilised by Galileo in his "Dialogues Concerning Two New Sciences"[14] . Galileo imagined a solid beam to be composed of parallel fibres, or filaments, effectively independent, the total tensile load carried by the beam being the sum of the tensions in each filament. What must be stressed is that although Galileo talked in terms of beams, which often are mage of fibrous material (wood), his discussion was intended to apply to beams of any solid material, so that the fibres are truly fictions. In fact Galileo mentioned stone beams in his discussion. Galileo used "Fibre" skilfully and was probably aware of such caveats to be attached to this heuristic as that must check whether it s a reasonable first approximation to consider the fibres independent.

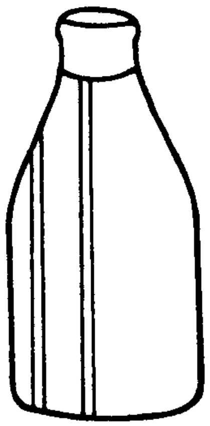


Fig (viii) Columns A
Showing the two fluid columns discussed

The heuristic "Column" is very closely related to "Fibre". One might say it is merely "Fibre" applied to fluids, so that it is the very same heuristic. "Column" suggests that one analyses differences in fluids by considering the body of the fluid to be made up of cylindrical columns. The caveat of non-interference between adjacent fibres/columns is still relevant here. In the next sub-section we will discuss further the issue as to whether Fibre/Column are two heuristics or one. For the moment, let s consider a particular "Column" approach to *Milko*. We ll present not only a successful solution route along which "Columns" will pull a Statics Algorithm - - but we ll also note one of the cul de sacs.

Consider the two fluid columns shown in Fig (vii), one near the axis and the other well off the axis. This suggests a bug - - it appears at first that the pressure must differ along the base of the milk bottle - - as the two columns are of different height. However this bug arose by ignoring wall pressure. By considering the static equilibrium of a horizontal fibre (column) of fluid it is possible to convince oneself that in fact there is a unique base pressure. It remains much easier then to consider a column about the axis of the bottle. To calculate the base pressure, there are two cases:

a) Contents homogenous: base pressure p
b) Contents stratified: base pressure p

Consider central columns, on base δA, in the two cases. The weight of the contents of column (b) is less than of column (a) – as basically (b) has an excess of cream. Expressing this evaluation algebraically,

$$p\delta A > p \; \delta A$$

or

$$p > p$$

That is, the base pressure decreases after separation. At this stage, one might return to examine the fine detail re the two columns a and b to realise that we have ignored side forces: no matter if sides are vertical as these forces don t contribute to the sums considered. In fact the prime heuristic message to be leaned from this calculation could be summed up in the following heuristic:

A *Select a thin vertical column that does not intercept any sidewalls*

A is one of the Problem Reduction/Algorithm Selection Devices associated with the "Column" heuristic.

Columns (Reduction Device B)

We ve already suggested that the preceding application of "Columns" amounted to an application to a hydrostatic context of the "Fibre" heuristic . However, the special convenience of the central column of Fig (viii) is not its thinness, but that having vertical sides, the thrusts on the walls of the column had no vertical components. So that it s very natural to consider columns of very large cross-sectional area in hydrostatics. All will do well, unless the column chosen hits a slanting wall. This is a bit of a nuisance (bug), but there is a way out as detailed below. But in debugging "Columns" to motivate a solution like that presented below - - the connection with "Fibre" is getting a little remote. Thus one should say that originally "Column" was just a portion of the heuristic "Fibre", but ultimately with elaboration (debugging) it assumes autonomy as an independent heuristic - - possessing a core common with "Fibre". This is a very important process in intellectual development that I call replication of heuristics: the mother heuristic spawns a daughter with many common elements. However, the idea of replication is part of my more elaborate psychological model of problem solving – and its presentation I do not see as part of the teaching stratagem I espouse. Certainly if the sort of application of "Columns" presented below is as far as this heuristic is elaborated, the solution given is still reasonably conceived as motivated by "Fibre" debugged for hydrostatics.

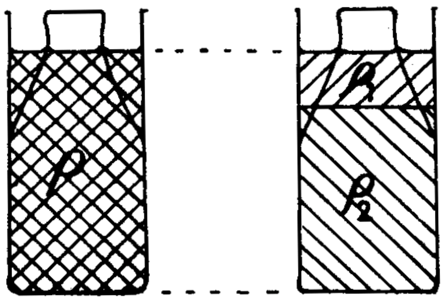


Fig (ix) Columns B Approach to *Milko*

Let us make the thought experiment of enclosing the milk bottle in a cylinder, sharing the same base, as in Fig(ix). Our aim is to reduce the *Milko* problem to a discussion of the pressure at the base of columns standing on the base of the milk bottle. Suppose that in the case when the milk bottle contains homogenous milk, the space outside the bottle, but in the cylinder is filled with milk to the same level as within the bottle: the volume of milk exterior to the bottle we call V_{ext} . Likewise. in the case when the milk has separated into components of density ρ_1(cream) and ρ_2 (creamless milk), suppose the exterior volume V_{ext} within the cylinder is filled to corresponding levels with cream and creamless milk (See Fig (ix)). The presence of the sloping neck of the bottle stands in the way of a "Columns" motivated algorithm, but we can justify ignoring its presence. Since pressure depends on depth alone, the pressure of each side of the bottle is the same, so that the pressure at the base of the bottle, p (for homogenous milk), p for stratified (separated)milk)) is unaffected if one removes the bottle walls, but leaves the fluid contents just as they were. Then considering the static equilibrioum of the vertical columns standing on the base aea A of the bottle one has

$$pA = W + \rho V_{ext} g$$

$$p \;\; A = W + \rho_1 V_{ext} g$$

In these equation W is the weight of the contents of the bottle, $\rho_1 V_{ext} g$ is the weight of the fluid in the exterior volume in the case where this fluid is predominantly cream. Hence we see at once that

$$p > p$$

In summary, the significant driving motive in producing the above derivation is the "Column" heuristic - - relentlessly applied to enable consideration of a vertical column of fluid standing on the bottle base area. This is a striking example of a more sophisticated problem reduction: bringing to light a Problem Reduction/ Algorithm Selection Device which we denote by B, roughly as follows:

> B*: Choose a vertical column with an "interesting" base. Remove intersecting walls whilst retaining fluid equilibrium.*

Add Effects

The heuristic "Add Effects" encapsulates the idea of (independent) causes having an additive cumulative effect. A verbal formulation of this problem solving schemata would be:

If X causes effect E,
and Y causes effect F,
then X+Y causes effect E+F

To implement "Add Effects" in a given problematical situation one must devise or select quantities that can meaningfully be added together[15] . In fact one aspect of the evolution of the field concept, and vector and tensor notation, of classical electromagnetic theory was the devising of a formalism in which "Add Effects" was more or less "built-in", as is especially exemplified by the "principle of superimposition" for fields. Likewise "Add Effects" is explicit in various additivity rules and implicit in the formalism of all those theories of physics characterised as linear. It is an enlightening struggle to make an "Add Effects" foray at the Dragon *Milko.*

In "Layman s Physics" it s the cream and milk minus cream (which we glibly term water) which "cause" the pressure at the base of a milk bottle. A little more formally, if the effect is additive, one would write

$$p = p_{cream} + p_{water}$$

and a like expression for the base pressure after separation, p . Now the total amount of cream is unchanged after separation, so that if quantity alone determines pressure, then

$$p_{cream} = p_{cream} \quad \text{(false !)}$$

and likewise

$$p_{water} = p_{water} \quad \text{(false !)}$$

leading to the canonical wrong answer, p = p . A more recondite, and equally false, version of this argument recalls that the total pressure of a gas mixture is the sum of the partial pressures of the components, so that on (mis)treating the components of milk as gases, one deduces a strict additivity of effect as above.

The obvious bug in the above discussion is that distribution must be taken into account. For the moment, we simplify the discussion by only dealing with the regular shaped "mathematician s milk bottle". Then in accord with "Add Effects", one envisages milk as the superposition of cream of density = $\rho_1 V_1 /(V_1 + V_2)$ and of milk minus cream = water of density $\rho_2 V_2 /(V_1 + V_2)$, both cream and water being dispersed throughout the total volume $V_1 + V_2$. Then the two additive components of pressure before separation are:

$$p_{cream} = \rho_1 V_1 (V_1 + V_2)^{-1} (h_1 + h_2)g$$

$$p_{water} = \rho_2 V_2 (V_1 + V_2)^{-1} (h_1 + h_2)g$$

So that by "Add Effects"

$$p = p_{cream} + p_{water} = (\rho_1 V_1 + \rho_2 V_2)(V_1 + V_2)^{-1} (h_1 + h_2)g$$
$$= = (\rho (h_1 + h_2)g$$

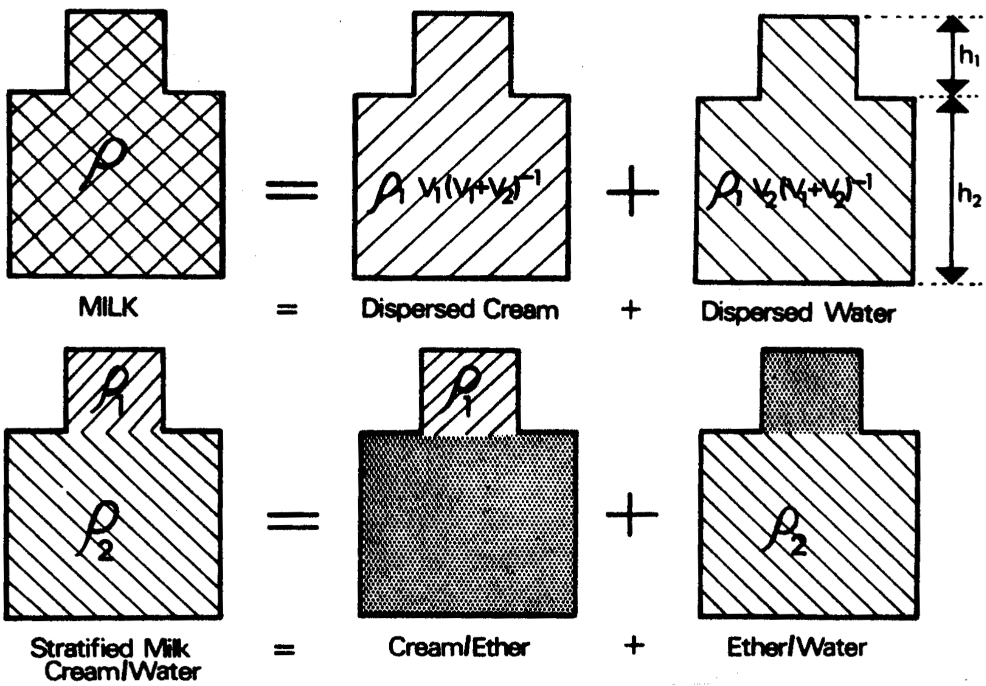


Fig (x) "Add Effects" decomposition applied to the "mathematicians milk bottle".

The heuristic has worked beautifully for milk. However, when we turn to calculate via "Add Effects" the base pressure after separation, we run into the super bug mentioned in Section 1. To implement "Add Effects" one needs to imagine that (as is shown in Fig (x), a "pressure ether" of zero density that fills up empty spaces, and transmits pressure so that one can calculate the new component pressures as:

$$p_{cream} = \rho_1 h_1 g$$
$$p_{water} = \rho_2 h_2 g$$

Thus

$$p_{cream} - p_{cream} = - \rho_1 (V_1 + V_2)^{-1} (V_2 h_1 - V_1 h_2)$$
$$p_{milk} - p_{milk} = \rho_2 (V_1 + V_2)^{-1} (V_2 h_1 - V_1 h_2)$$

If $V_1 = h_1 A_1$ and $V_2 = h_2 A_2$

and the simulation of a milk bottle entails $A_1 < A_2$

Then $(V_2 h_1 - V_1 h_2 = h_1 h_2 (A_2 - A_1) > 0$

Thus as water is denser than cream, we have

$$p - p = (p_{cream} - p \ _{cream}) + (p_{water} - p \ _{water})$$

$$= (\rho_2 - \rho_1)(V_1 + V_2)^{-1}(V_2 h_1 - V_1 h_2) > 0$$

Thus, for a "mathematical milk bottle", we have established that the pressure near the base, p drops to the value p after separation of cream. Presentation of this more sophisticated derivation to students leaves for them the more general puzzle of extending this derivation to milk bottles of conventional shape. In fact the argument given above applies at once to a conventional bottle provided cream/milk volumes and vertical heights satisfy the inequality

$(V_2 h_1 - V_1 h_2) > 0$

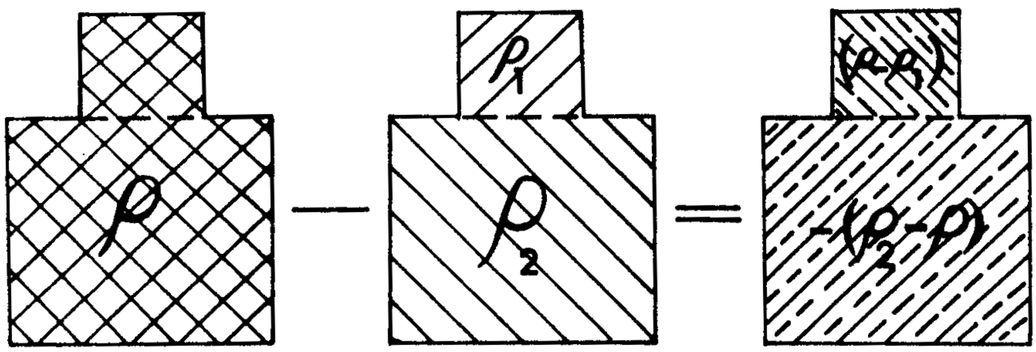
i.e.,

$$V_2/h_2 > V_1/h_1$$

which is a requirement on the average cross-sectional areas.

Subtract Effects

This <u>heuristic</u> is conceived by this writer as a variant of "Add Effects" . A "Subtract-Effects" motivated calculation of the differences in base pressure, p-p , is outlined in visual terms in Fig (xi). Now in this figure we have not introduced a "pressure-ether" - - but the lower volume V_2 of the mathematical milk bottle in the right-hand side case contains a liquid of negative density!



p p p – p '

Fig (xi) Schematic outline of "Subtract Effects" motivated attack on *Milko.*

$$p - p = (\rho - \rho_1)_2 h_1 g + (\rho - \rho_2) h_2 g$$

It is then clear that p > p provided that the (positive) contribution to base pressure from a liquid of density $(\rho - \rho_1)$ in the narrow upper volume exceeds the contribution of liquid of density $-(\rho - \rho_2)$ in the lower volume.

Process

"Process" is a heuristic of great power which involves the notion of a state. From the "Process" viewpoint, a problem is conceived as devolving on a transformation, like so

(State A) → (State B)

or more briefly, A → B . In terms of the parameters that define a state, the transformation is

$$A \rightarrow B \quad \Leftrightarrow \quad (a_1, a_2, a_3, a_4, a_5, a_6 \ldots) \rightarrow (b_1, b_2, b_3, b_4, b_5, b_6 \ldots)$$

The key problem solving idea of "Process" is to devise some possibly fictitious state X, for which the transformation rules for

A → X ; X → B

are well established, so that one can readily compute the transformation of parameters,

$$(a_1, a_2, a_3, a_4, a_5, a_6 \ldots) \rightarrow (x_1, x_2, x_3, x_4, x_5, x_6 \ldots) \rightarrow (b_1, b_2, b_3, b_4, b_5, b_6 \ldots)$$

What has presented above is a very sophisticated and formal description of "Process". In fact the present writer first identified this heuristic as being potent in thermodynamics and special relativity and conceived of this problem solving idea as being used and developed only by advanced students. However, in September 1974, I was flabbergasted to observe a five year old child, Leo, use this same heuristic. At the conclusion of a classic Piagetian interview described in the Appendix, Leo was asked:

> *"How would you explain to another child why the Pepsi (poured from a squat beaker) rises so high after pouring (into a narrow cylinder)?*

Leo thought intently for a few seconds, then answered,

> *"The sides are pushing the Pepsi up"*

Leo placed his hands apart and forward, the brought them together as he said this. It was clear in context that he had invented a fictitious

State X Pepsi in the tall cylinder had the same diameter as the (squat) beaker

In State X the cylinder and would hold its aliquot of Pepsi at the same level as that in the beaker. Leo s explanation entailed the transformation from

State A: Pepsi in squat beaker

to the final state

State B: Pepsi in tall narrow cylinder

via the fictitious state X.

To return to a "Process" analysis of the Dragon *Milko.* where one perceives this Dragon as involving a transformation from

State A: Homogenous milk in milk bottle.

to

State B: Stratified milk in milk bottle.

One can t compute the alteration in base pressure - - i.e., $p_A - p_B = p - p$ directly - - after all, this is the problem of this Dragon. Yet if the neck of the milk bottle was rubber, or was hinged somehow, and the bottle transformed into a cylinder it would be easy, in fact trivial, to compute the base pressure change after stratification by reference to the states:

State X: Homogenous milk in cylinder

State Y: Stratified milk in cylinder

In a cylinder the only vertical forces acting on the fluid contents (of total weight W) are gravity and the base pressure acting over the area A, so that

$$p_X = p_Y$$

The additional base pressure in State A compared to State X is due to an additional height D of milk so that under the transformation

$$A \rightarrow X: \quad p_A - p_X = p - W/A = D\rho g$$

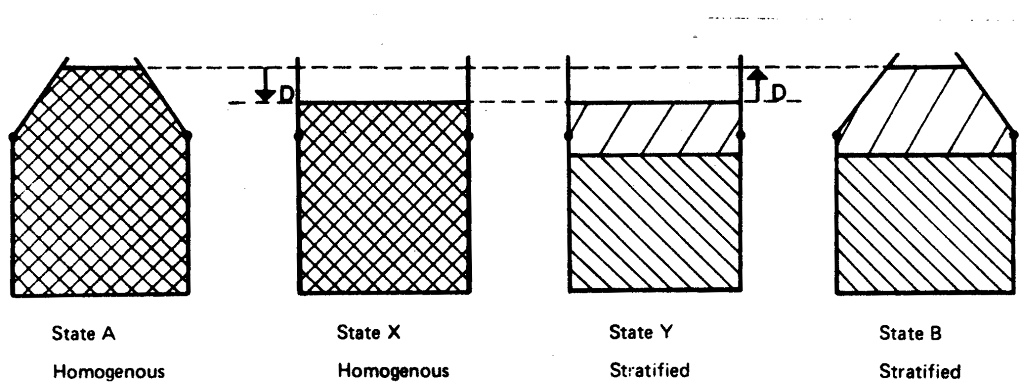
Likewise

$$Y \rightarrow B: \quad p_Y - p_B = W/A - p = D\rho_1 g$$

Hence

$$p - p = p_A - p_B = D(\rho - \rho_1)g$$

which is positive as cream density ρ1 is less than the density ρ1 of milk. This "Process" argument is illustrated in Fig (xii)



Its worth noting an unsuccessful "Process" motivated attack on *Milko* that a number of students initiate. Suppose the milk bottle is connected near its base with a vertical cylinder, as drawn in Fig (xiii) below. The level of homogenous milk is equal in the two branches at the initial state. Subsequently the milk stratifies; however there are unequal lengths of strata in the two connected vessels, and there is no convenient intermediate state.

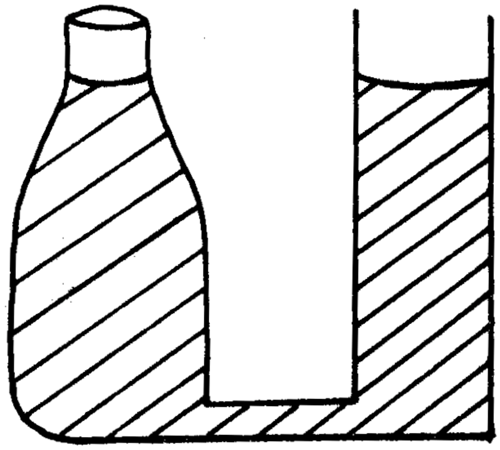


Fig (xiii) Sketch for an unsuccessful "Process" foray at *Milko*

Its clear that students attacking *Milko* in this way recognise that in a (closed) cylinder pressure will be unchanged after separation, so that the cylinder holds something like our States X and Y. However the application of "Process" is essentially incomplete, as there is no actual transformation between initial state and X, and final state and Y.

JUGGLO

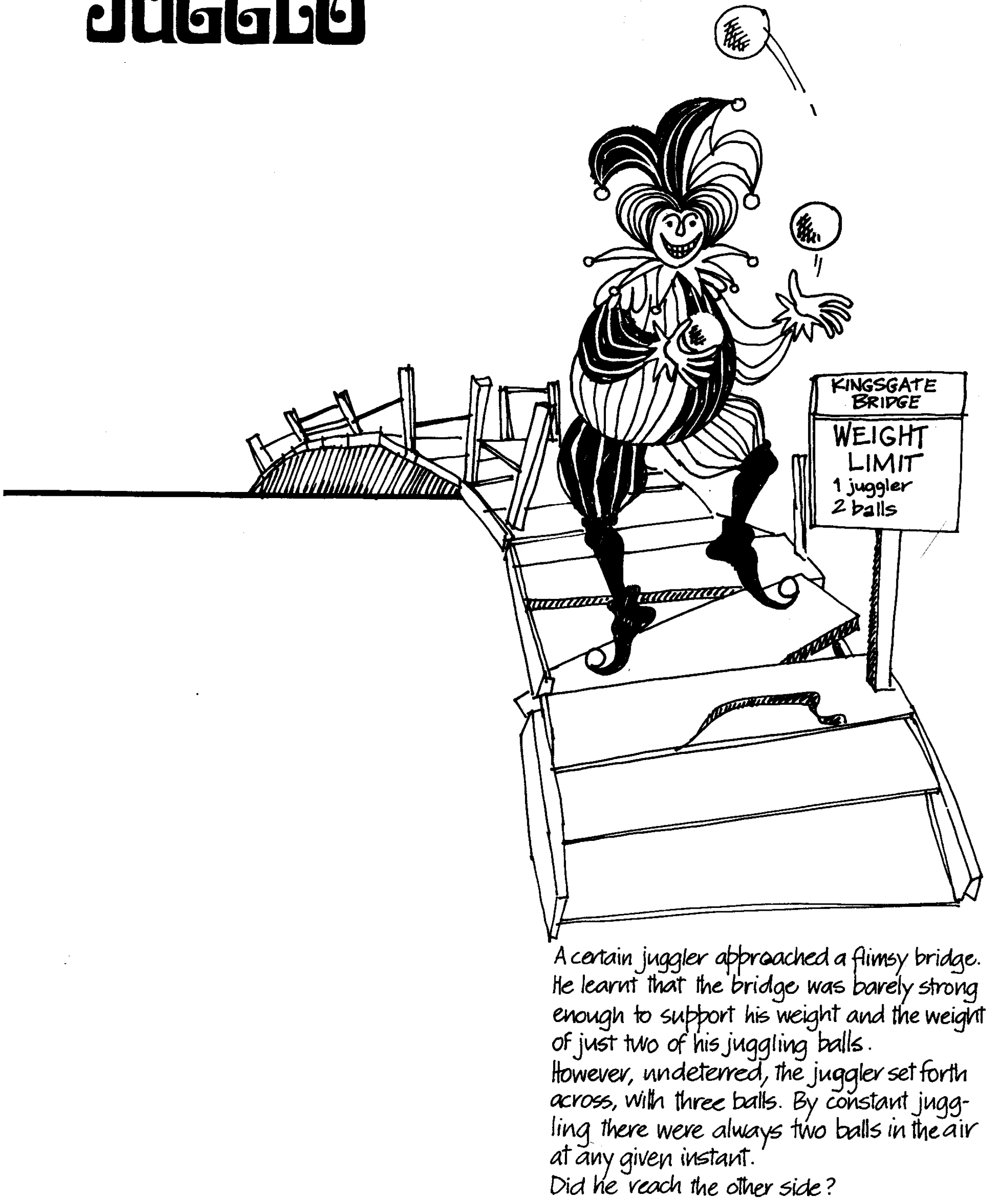


Fig (xiv) The Dragon Jugglo reproduced with permission from Harvey A. Cohen, "A Dragon Hunter's Box", Hanging Lake Press, Warrandyte, Victoria 3113, Australia

3 JUGGLO

Prelininary

The Dragon *Jugglo* of Fig (xiv) is a superlative Fermi problem that appears to have been first posed sometime in the nineteenth century. On being exposed to this Dragon a typical La Trobe undergraduate will firmly answer "No!" With some coaxing the student can be encouraged to recount the thinking that lead to this conclusion. A typical response - - very much refined for these didactic purposes - - goes like so: "The bridge has a safe load ofMg + 2mg and presumably will collapse if this load is exceeded. It's supporting a Juggler and three balls of total weightMg + 3mg. The balls are in the air sometime, and the juggling details are probably horribly complicated - - but the real point is that the whole system (= juggler + 3 balls) of a weight which exceeds the critical load - - so that the bridge collapses."

The basis for this correct "physical intuition" - - the response "No" as revealed by such verbalising - - lies in the mechanical implications of the heuristic "In Toto".

However, if the tutor reformulates *Jugglo,* supposing that there are only two balls in all which the juggler is tossing on the very same bridge, incorrect solutions are common, if not so invariable as in the case of *Milko*.

Now to get down to the slaughter of *Jugglo*. Here are four different attacks - - each named in accord with its dominant heuristic.

In Toto

A familiar application of this problem solving schema is to the description of rigid bodies where the concept of the centre of mass is introduced. To apply this idea to *Jugglo* involves considering the system of Juggler (of mass M) and N balls each of mass m as a single body of mass M + Nm.

Students often adopt an "In Toto" viewpoint to examine *Jugglo* - - but in midstream switch heuristic – following the flag

(c) Switch to Divide and Conquer Viewpoint

It seems that there is a particular debug routine attached to the "In Toto" frame

(d) Check the relation of the parts to the whole

that is easily confusable with having switched to "Divide and Conquer". In fact we use such a debug routine (d) to extend the fairly crude "In Toto" argument given above to the following polished attack on *Jugglo*.

How do the component parts of the whole *Jugglo* system interact? The answer is reassuring to the "In Toto" champion: the "internal" forces between the components are equal and opposite, and therefor of no consequence in considering the motion of the system in terms of the behaviour of the centre of mass. The various "external" forces, including the {upward) reaction of the bridge R, have a sum of magnitude

$$R = (m + Nm)g$$

and R is in the vertical direction. During juggling, the centre of mass of the system moves up and down a little, about some average position (or perhaps remains stationary). Consequently, if at any instant the centre of mass is experiencing an upwards acceleration, then at that instant

$$R > (M + Nm)g$$

Thus even in the case of two balls (N = 2), if the centre of mass of the system comprising the Juggle and balls is not stationary, then at some instant there will be a net upward acceleration and the bridge load limit will be exceeded.

Divide and Conquer

A "Divide and Conquer " approach to a problem is to break the problem into interfacing problems, each of which is solved in turn. Applied to *Jugglo* this heuristic would naturally lead us to consider separately the dynamics of the bridge, the juggler, and each of the three balls. Now the bridge is specified as capable of supporting a maximum load of (M + 2m) g, M being the mass of the juggler, and m the mass of the ball. The first sub-problem - - the juggler - - is easily analysed to deduce that the maximum force that the juggler can exert on one (or more) balls at any instant is 2mg upwards. The next sub-problem is the motion of one ball, ball I say. If at time t – 0 the ball is released with upward velocity v it will rise a distance v/2g in time v/g, and after a time lapse of 2v/g it will return to the altitude of release, but now with downward velocity v. If caught at the same height as when released, then (presuming the juggler has no other balls in his hand at the time) the juggler can apply (maximum) upward force 2mg on the ball, so that the net force on the ball is mg upwards - - leading to a symmetric reversal of the motion as per Fig (xv).

Fig (xv) Possible motions of two balls tossed by the juggler

In Fig (xv) the altitude of ball I over time is shown, where release and capture occur at constant height, together with a consistent motion for the second ball (dashed). Clearly in accord with this analysis at all ttimes the juggler is applying the maximum allowed force so that there is no possibility of him catching a further ball; there can be no Ball 3 without exceeding the bridge load limit.

The chief virtue of this "Divide and Conquer" attack is the very detailed information derived as to an acceptable juggling style for two balls: if the greatest height reached by a ball was h (= $v^2/2g$) above catching level (marked 0 in Fig (xv)), the ball will fall a further h encased in the juggler's hand and then be brought up to be released at the catching level whilst simultaneously the second ball is caught - - possibly with the other hand at a different altitude - - after the second ball has likewise fallen through h.

Divide and Conquer ("Time Average" Algorithm)

This approach to *Jugglo* is also motivated by "Divide and Conquer". However the trick of taking a time average (such as is often done in statistical mechanics) is used to get rid of uninteresting dynamical detail.

Consider the vectorial equation of motion for ball a:

$$m(d/dt)\mathbf{v}_a = -mg\mathbf{k} + \mathbf{F}_{aj}(t)$$

where we denote vector quantity in bold, k is the unit vector in the vertical direction, and $\mathbf{F}_{aj}(t)$is the force applied by the juggler to ball a at time t. Integrating betweeen the limits t = 0 to t = T

$$m\,\mathbf{v}_a(T) - m\,\mathbf{v}_a(0) = mgT\,\mathbf{k} + \int_0^T \mathbf{dt}\,\mathbf{F}_{aj}(t)$$

Hence, the time-averaged value of the force $\mathbf{F}_{aj}(t)$ is

$$< \mathbf{F}_{aj} > = mg\mathbf{k} + (m\,\mathbf{v}_a(T) - m\,\mathbf{v}_a(0))/T$$

Provided that this ball isn't dropped, the numerator of the second term on the left is bounded, so that after an extensive duration of time,T, the average value of $\mathbf{F}_a$ is

$$< \mathbf{F}_{aj} > = mg\mathbf{k}$$

Summing the forces on the jugglrer, and then considering the load on the bridge, gives for the time-averaged load on the bridge in the case of three balls:

$\langle \mathbf{R} \rangle = Mg\mathbf{k} + 3mg\mathbf{k}$

which exceeds the prescribed limit. For two balls

$\langle \mathbf{R} \rangle = Mg\mathbf{k} + 3mg\mathbf{k}$

Hence, if, and only if**,** **R** is constant., the bridge is safely loaded. But if **R** varies then at some instant its magnitude must exceed the average value.

Add Effects

In accord with the heuristic “Add Effects” we conceive the load on the bridge as being the cumulative (additive) effect of each of the juggler, and three balls all separately contributing. Thus the bridge is “held responsible” for on the average keeping these four objects above the bridge’s walkway. The uggler needs a vertical force of magnitude Mg to stay more or less where he is, and likewise each ball requires an external force of average mg (vertical upwards) . Hence the safe load is exceeded by a juggler tossing three balls.

The informal discussion just presented differs in small but crucial emphases from that given under the heading of “In Toto” A formal mathematical argument motivated by “Add Effects” is likewise parallel with that presented under “In Toto”.

Incomplete Section

Due to time constraints, not all of this section of MIT AI Lab Memo 338 has been digitised and optical reading errors corrected..

Try again at a later date.

CONCLUSION

We have shown how a diversity of "solutions" to the Dragons *Milko* and *Jugglo* depend on just a limited number of problem solving schemata called heuristics. The core idea of these heuristics is probably derived in childhood, but during intellectual development a coterie of debug routines, caveats, flags, problem transformation and reduction ideas become attached to each heuristics . Knowledge of very specific skills termed algorithms is also linked with particular heuristics .

Consistent with this analysis, a teaching stratagem is outlined that aims to promote student self-awareness of the processes involved in their own intellectual development, and of the evolutionary character of the formulation of the solution to a formidable problem. That is, the teaching stratagem aims to teach how to find solutions rather than to teach solutions., by exposing students to a model for problem solving whilst at the same time exposing them to challenging problems, in particular Dragons.

ACKNOWLEDGEMENTS

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Appendix

This paper as a whole has been concerned with the development of problem solving ability in physics. However the teaching stratagem I espouse is based on a theoretical model of intellectual development that has far greater gambit. In this Appendix the model is applied to give an explication of certain aspects of the intellectual development of children, by showing how it interprets some of the data obtained in the "protocols (transcripts) of three Piagetian experiments.

The Egg and Egg Cup Experiment
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In order to answer questions as to whether some quantity is greater or less than another, the typical child uses such heuristics as:

H1: Greater extent means more.
H2: Sparseness (greater gaps between elements) means less.
H3: Counting tells you if more or less.

The heuristic H3 is only suitable for very small sets because of a (young) child s limited skill at counting. The sorts of situations where a typical child of five years gives the correct answer to questions about quantity are shown in Fig (xvi), A being what we term a paradigm for H1, while B gives two paradigms for H2. It is a notable about these paradigms that only one heuristic is applicable to each. But what happens if a situation is presented to which both heuristics are applicable, and give conflicting conclusions? In one of the classic "conservation" experiments of Jean Piaget, the egg-cup experiment, children in the 4-7 years age group are set such a clash in their heuristics. As indicated in Fig (xviC) if such children are shown a line of eggs **in** egg-cups, where the extent and sparseness of both the eggs and the egg-cups are the same, then in answer to the question "Are there more eggs or more egg cups?" the typical child (4 – 7 years) answers "no the same." However, if the set-up stays in full view of the child while the eggs are removed from the egg cups and spread out in a longer line than the line of the cups – then the situation is one in which H1 and H2 give conflicting assessments to the repeated question. However, for the young child, H1 is in some way tagged as primary or more important - - for, as indicated below, H1 describes a great range of situations where such evaluations are sought. So the typical five year old will now reply, "More eggs". In contrast the seven year old will give the adult answer "Of course not." What distinguishes the seven year old from the typical five year old? Possibly the seven year old has acquired a heuristic such as:

H4: Relationships "more than" or 'less than" remain true if items are moved but not removed

However, the mere addition of H4 to a child s repertoire won t necessarily lead to the correct answer to the repeated question of the egg and egg-cup experiment. What is needed is some caveat like

H5: In case of conflict between H1 and H2 use H4

The addition of these - - or somesuch - - heuristics to the <u>heuristic</u> that holds the collection H1, H2, etc of the typical five year old child is an instance of what I term the debugging of **heuristics**.

Another Piagetian 'Conservation" Experiment

Here is the protocol of classic Piagetian "conservation" experiment, conducted by one of Piaget and Inhalder s collaborators, Olivier de Marcellus, in Lexington, Massachusetts in September, 1974.

A five year old child Rob [name altered] was shown two vessels. One, a measuring cylinder, was tall and narrow in cross-section, the other was a squat beaker containing a dark liquid termed "Pepsi". Rob was asked to what height he anticipated the "Pepsi" poured from the beaker would fill the narrow cylinder.

Rob pointed to a level on the cylinder at the same height *(1) as the top level of the "Pepsi" in the squat vessel. The "Pepsi" was poured into the cylinder. The level in the narrow cylinder was about three times higher than that predicted by Rob. Rob registered much astonishment, followed by a traditiona;

facial expression for grasping a tricky idea. Rob was asked: "Is these more Pepsi now?". Rob replied, "No! It s just the same . . . . it only looks more". *(2).

Rob was then asked how he would explain to another child how it was that "Pepsi" was so high in the (narrow) cylinder. Rob pondered a moment - - then placed his hands about 20 centimeters apart in front of him, then steadily drew his hands together while saying, "The sides are pushing the Pepsi up" *(3). Rob s responses *(1), *(2), *(3) of the above protocol, merit these comments:

*(1) Rob s expectation opf the height of the new (narrow) Pepsi column conforms to the heuristic H1 of the preceding experiment. The anticipated extent of the new Pepsi column - - its height - - was anticipated to be unchanged.

*(2) Rob opined a caveat to be referred to as H6 which he probably only recently learnt to associate with the heuristic H1.

H6: Sometimes it only looks more.

- Rob s response "The same" was undoubtedly guided by heuristic H4 or some similar historical heuristic. Rob was clearly on "the cusp" of becoming a conserver in Piagetian terms.

*(3) Rob had formulated an explanation in terms of the heuristic Process - - the same heuristic, which somewhat elaborated was used to snare the Dragon *Milko* in Section 2. Rob was considering a fictitious state of the cylinder - - presumably one in which the cross-section was the same as in the squat beaker. In the fictitious state, the Pepsi would be at the same level as in the squat beaker. But on bringing the sides closer together - - as indicated by Rob s hand movement - - the Pepsi level would rise.

Islands Experiment

The following incident took place within the context of a very extensive Piagetian experiment, "Islands", conducted by Seymour Papert.

A five year old child was asked to count the thirty six (2cm x 2cm x2cm) cubes arranged as a rectangular prism which was 8cm x 6cm x 6cm in size. Her algorithm was transparent, as she traced her finger row by row along the front face, and proceeded to likewise count blocks on other faces of the prism. She concluded there were 30 cubes in the prism. She was asked, "How did you do it ? If another child wanted to count the blocks, what would you tell her ?" The child replied, "Don t count the side [=edge] ones twice". The child failed to say that her basic method was the use of H3 in systematically tracing her finger along the faces. – and had described her procedure in terms of the bug in that method of which she was now very aware. (Of course her counting failed to count the inner blocks).

The point being made is that in problem solving us, the solver may often take for granted the overall heuristic , and lay emphasis on aspects of the heuristic that required refining or debugging. For instance, in applying the heuristic "Add Effects" to *Milko* in Section 2, a rather bizarre artifact, a pressure transmitting ether had to be introduced for this heuristic to succeed. Yet it would be patently misleading to characterize this solution as the "The Pressure-Ether Model" for the *Milko* Dragon.

Interpretation of "Conservation" in terms of Heuristic Frames

In describing above some classic Piagetian "Conservation" experiments[16] we have noted the heuristics manifestly utilized – and in some instances verbally expressed by children in the five to seven years age group. Perhaps we should note that it is fairly novel to attempt to use the protocols of such experiments to determine the heuristic repertoire of a child: such a discussion was first given by Seymour Papert[17]. The evidence of these and other protocols suggest that a child does not mature by discarding the "non-conserving" heuristics and learning a more precise "conserving" heuristic: rather to the prototype heuristic "To tell if more - - look" are added further structural elements - - other heuristics - - the whole collection of heuristics being closely linked, and heuristics relating the various elements are part of the whole. Table II shows how some of the heuristics discussed above slot into the heuristic which is called "Look – More".

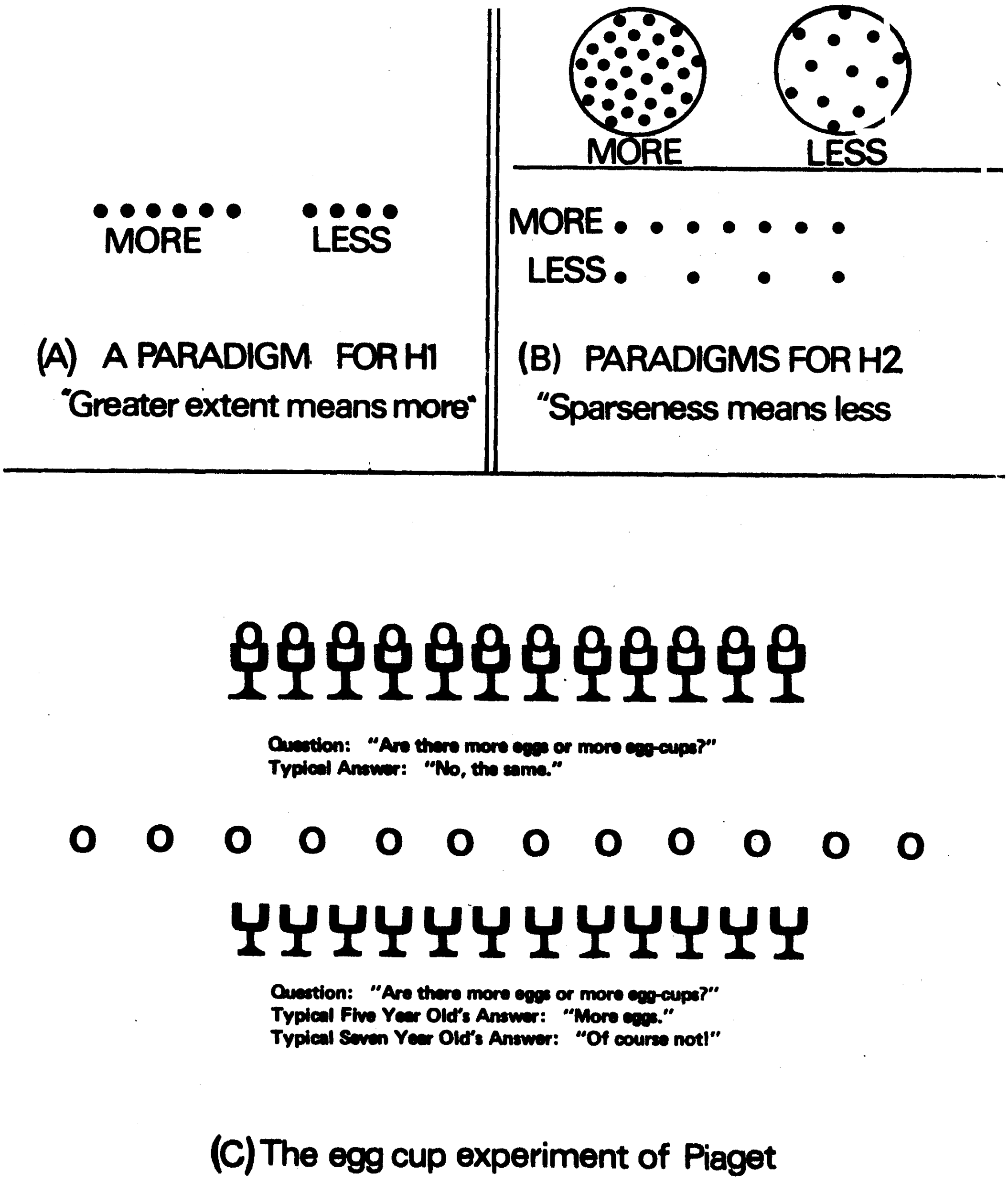

Fig (xvi) Figures relating to "conservation" protocols. See Text

TABLE II

THE ANATOMY OF THE <u>HEURISTIC</u> "LOOK-MORE"

COMPONENT	SPECIFICATION
Core Heuristic	"To tell if more – look"
Problem Reduction	H1: "Greater extent means more"
Devices and Algorithm	H2: "Sparseness means less"
Selector	
Debug Routines	"Check H1 and H2 for consistency"
Demons	H5: "In case of conflict between H1 and H2, use an historical heuristic. H6: "Sometimes it only looks more"

The heuristic detailed in Table II is similar to a schema proposed by Minsky and Papert[18]. The young child has available the core heuristic of this frame - - the idea that visual observation can be used to determine quantity - - plus H1 and possibly H2. Of course to a young child quantity means capacity to satisfy hunger or maybe number of bites to consume. One of the most endearing protocols I have collected was of a non-conserving six year old, who was asked whether a flattened ball of dough contained more than a spherical ball which had previously been adjudged "the dame amount". The girl guided by H1 claimed the flattened ball of dough contained more, and justified this answer by pointing out that the round ball could be eaten in two bytes, where the flattened ball would take five bytes. The older child - - the Conserver - - had added to these basic elements of a heuristic frame debug routines and and deons akin to those in the table. It is just that process of augmenting and editing a frame, such as "Look – More" which is called in this paper the debugging of the heuristic.

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[7] G. Polya, *How to Solve It*, 2nd edition, Double Bay, Garden City, N.Y. (1957), *Mathematics and Plausible Reasoning*, Princeton University Press, Princeton, N.J. (1954).

[8] To claim that Dragons or any other problems have the flavour of research requires for justification a detailed discussion of the pattern of scientific progress. The philosophers of science, Hanson, Kuhn and Lakatos are especially relevant in this regard. See

N.R. Hanson, *Patterns of Discovery*, Cambridge Univ. Press, U.K., (1958)

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[9] The sort of solution that Fermi anticipated for this dragon would be something like so. "There are 8 million people in New York, 80% say of pianos are in family homes and apartments, of which one can estimate . . if there is one such unit per 5 people as 1.6 million. Such and such a fraction of homes possess a piano. A piano needs tuning after such a period.. . . Piano owners perceive a piano needs tuning after a further time-lapse, or when certain pitches are significantly in error. To tune piano takes an estimated amount of time, so that to tune the requisite weekly number of pianos requires so many piano tuners working 35 hours per week, including travelling and administrative time . . . "

[10] H.A. Cohen, *What G Killed Ned Kelly? and Other Problematical Dragons*, La Trobe University, Melbourne, Australia (1972).

[11] H.A. Cohen, *Mathematical Dragon Hunting on the La Trobe Campus*, Australian Vice-Chancellor's Committee Educational Newsletter, No3/73, Published by the A.V.C.C., Canberra, ACT, Australia, 1973.

[12] In his note, entitled *Painless precession*, Eastman gave just such a "Divide and Conquer" motivated approach to precessional problems. This note also lists previous American Journal of Physics discussions of this 'mystifying' phenomena. See P.C. Eastman, Am. J. Phys. **43**, 366 (1975).

[13] Professor Feynman has confirmed the substance of this account in a private communication.

[14] See p. 148, Galileo Galilei, *Dialogues Concerning Two New Sciences*, translated by H. Crew and A. de Salvio, Dover, N.Y. (1954).

[15] This point is discussed from a philosophical perspective in Chapter 7, entitled *Extensive Magnitudes* in Rudolph Carnap, *Philosophical Foundations of Physics*, Edited by Martin Gardner, Basic Books, (1966).

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[18] Marvin Minsky and Seymour Papert, *Progress Report* MIT Artificial Intelligence Laboratory, Artificial Intelligence Memo No 252 (1972). See Section 4.1