The C-Conserving Decay Modes $\eta \to \pi^0 e^+ e^-$ and $\eta \to \pi^0 \mu^+ \mu^-$

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The C-conserving decay rate for $\eta \to \pi^0 e^+ e^-$ is of second order in the fine structure constant. A knowledge of the matrix elements $\eta \to \pi^0 \gamma\gamma$ and $\gamma\gamma \to e^+ e^-$ is sufficient to find the imaginary part of the matrix element $\eta \to \pi^0 e^+ e^-$ from unitarity. The real part of the matrix element can be determined from dispersion relations. Recent theoretical estimates(1) of the decay rate for $\eta \to \pi^0 e^+ e^-$ have assumed an interaction $\eta \pi^0 F_{\mu\nu} F_{\mu\nu}$ for the $\eta \pi^0 \gamma\gamma$ amplitude(2). However, this interaction leads to the result that the decay rate into one π^0 and a lepton pair is proportional to the square of the mass of the leptons. The rate is therefore very much larger for muons(3) than for electrons, whereas the contrary conclusion is reached using the simple phase space argument (which supposes a constant matrix element). One way to modify this result is to note that the C-conserving decay amplitude is decomposible into two invariant amplitudes: the first corresponding precisely to the S-wave interaction given above and the second to the P-wave interaction included in $\eta \pi^0 P_\alpha P_\beta F_{\alpha\mu} F_{\beta\mu}$ where P is the η four momentum. In this paper we adopt the second interaction leading to a calculation that is parallel to the calculation(3) by the second author based on the first interaction. We calculate both the $\eta \to \pi^0 e^+ e^-$ and the $\eta \to \pi^0 \mu^+ \mu^-$ rates. Chang(4) has performed a calculation using a vector meson intermediary, which corresponds to using the non-local interaction $\eta(x)\pi^0(y)[\delta_{\alpha\beta} F_{\mu\nu}(x) F_{\mu\nu}(y) - 2F_{\mu\alpha}(x) F_{\mu\beta}(y)](\partial/\partial x_\alpha)(\partial/\partial y_\beta)\Delta_F(x-y)$, where $\Delta_F(x)$ is the Feynman-Stueckelberg scalar propagator for the mass of the intermediate meson; he calculates the decay width for $\eta \rightarrow \pi^0 e^+ e^-$ after neglecting all terms proportional to the electron mass, so that this calculation is not extendable to the $\eta \rightarrow \pi^0 \mu^+ \mu^-$ decay. due to the large muon mass.

Before giving the details of our calculation, we would like to comment on the possibility of observing such decay modes. The present data[5] give $\Gamma(\eta \rightarrow \pi^0 e^+ e^-)/\Gamma(\eta \rightarrow \text{all}) < 3.7 \times 10^{-4}$. At the time of writing this note there seem to be indications that the rate for $\eta \rightarrow \pi^0 2\gamma$ decay[6] is smaller than had been reported previously[7]. A calculation of the C-conserving decay is best given in the form of a branching ratio with respect to the $\eta \rightarrow \pi^0 2\gamma$ rate, so that the unknown coupling constant is divided out. If the rate for $\eta \rightarrow \pi^0 2\gamma$ is small then the branching ratio $\Gamma(\eta \rightarrow \pi^0 e^+ e^-)/\Gamma(\eta \rightarrow \pi^0 2\gamma)$ is much smaller. In view of this it is not worthwhile attempting a calculation of the real part of the matrix element $\eta \pi^0 e^+ e^-$. The real part is always divergent, $\eta \pi^0 F_{\mu\nu} F_{\mu\nu}$ gives a logarithmic divergence and $\eta \pi^0 P_\alpha P_\beta F_{\alpha\mu} F_{\beta\mu}$ a quadratic divergence. We limit ourselves here to a complete discussion of the imaginary part which is sufficient to set a lower bound on the values of the branching ratios $R_1 = \Gamma(\eta \rightarrow \pi^0 e^+ e^-)/\Gamma(\eta \rightarrow \pi^0 2\gamma)$ and $R_2 = \Gamma(\eta \rightarrow \pi^0 \mu^+ \mu^-)/\Gamma(\eta \rightarrow \pi^0 2\gamma)$.

Let us first consider the $\eta \to \pi^0 2\gamma$ decay rate where the interaction is $\eta \pi^0 P_\alpha P_\beta F_{\alpha\mu} F_{\beta_\mu}$. In the frame where the η is at rest and the polarization vectors are space-like the only term in the amplitude is

$$T = \frac{g}{M^4} (\epsilon_1 \cdot \epsilon_2)(P.k_1)(P.k_2)$$

where g is a dimensionless coupling constant. Adding a factor of two for the crossed diagram and summing over the polarization vectors gives the following expression for the decay rate Γ,

$$\frac{128\pi^3}{M} \Gamma(\eta \to \pi^0 2\gamma) = \frac{g^2}{M^8} \int_0^{(M-\mu)^2} ds \int_{-x(s)}^{+x(s)} dx[s^2-(y^2-4x^2)s + \tfrac{1}{2}(y^2-4x^2)^2]$$

where $$2My = M^2-\mu^2+s$$

$$x(s) = \frac{1}{4}\left\{[(M+\mu)^2-s][(M-\mu)^2-s]\right\}^{\frac{1}{2}}$$

i.e.

$$\frac{128\pi^3}{Mg^2} \Gamma(\eta \to \pi^0 2\gamma) = 0.65 \times 10^{-2}$$

When more data is available on the γ spectrum it will be possible to distinguish between the interactions used for this decay.

Next consider the mode $\eta \to \pi^o e^+ e^-$. Let $Q = p+q$ where p and q are the lepton four momenta. If l is the loop variable, the numerator of the Feynman diagram is

$$N = \bar{u}(p)(l.Pg_{\alpha\mu} - l_\alpha P_\mu)(Q-l).Pg_{\alpha\nu} - (Q-l_\alpha)P_\nu)\gamma_\mu[(p-l).\gamma+m]\gamma_\nu v(q)$$

After some Dirac algebra this reduces to

$$N = \bar{u}(p)[2mP.QP.l+2P.QP.l\gamma.l-2m(P.l)^2-2(P.l)^2\gamma.l$$

$$-2P.QP.p\gamma.l+2P.pP.l\gamma.l-2p.Q\gamma.PP.l+\gamma.QP.l\gamma.l\gamma.P$$

$$+2P.lp.l\gamma.P+2p.PQ.l\gamma.P-\gamma.PQ.l\gamma.l\gamma.P$$

$$+l^2(p.Q\gamma.P-2p.P\gamma.p+\gamma.P\gamma.l\gamma.P-2L.P\gamma.P)]v(q)$$

The terms involving l^2 are eliminated when one sets the photon lines on the mass shell to get the imaginary part. We refer the reader to reference 3 for a tabulation of the integrals over $l_\mu, l_\mu l_\nu$ and $l_\mu l_\nu l_\mu$. Finally we collect the terms into the following decomposition.

$$T = \bar{u}(p)\ (\underline{I}mA(s,t) + \gamma.P\ \underline{I}mB(s,t)\ )\ v(q)$$

where A and B are even and odd respectively under crossing $t \leftrightarrow u$, and

$$\frac{A}{2m} = \frac{\underline{T}mA(s,t)}{2m} = (P.Q)^2 \left(\frac{s^3+6m^2s^2+20m^4s-6m^2s^2V-24m^6V}{3s(s-4m^2)^3} \right)$$

$$-2P.pP.q \left(\frac{s^2+26m^2s-12m^2sV-12m^4V}{3(s-4m^2)^3} \right)$$

$$-P^2 \left(\frac{2s^2-20m^2s-3s^2V+24m^2sV-24m^4V}{12(s-4m^2)^2} \right)$$

$$B = \underline{T}mB(s,t) = 2P.(p-q) \left(\frac{s^2-m^2s-3m^2sV+6m^4V}{3(s-4m^2)^2} \right)$$

$$V = \left(\frac{s}{s-4m^2} \right)^{\frac{1}{2}} \left| \frac{s^{\frac{1}{2}}+(s-4m^2)^{\frac{1}{2}}}{s^{\frac{1}{2}}-(s-4m^2)^{\frac{1}{2}}} \right|$$

As a check one can verify that the expressions reduce to those given in reference 3 when one replaces $P_\alpha P_\beta$ by $\delta_{\alpha\beta}$. The individual terms are finite at $s = 4m^2$ providing a check that there are no infra-red divergences. Crossing has the effect of multiplying A and B by a factor of two. Hence the decay rate is given by

$$\frac{128\Pi^3}{M} \Gamma(\eta \to \Pi^o e^+ e^-) = \frac{d^2g^2}{16M^{12}} \int_{4m^2} ds \int_{-x(s)}^{(M-p)^2+x(s)} dx \, [2A^2(s-4m^2)-16mMxAB + \frac{B^2}{2}((s+M^2-\mu^2)^2-16M^2x^2-4M^2s)]$$

where

$$x(s) = \left\{ \frac{((M+\mu)^2-s)((M-\mu)^2-s)(s-4m^2)}{16s} \right\}^{\frac{1}{2}}$$

i.e.

$$\frac{128\Pi^3}{Mg^2} \Gamma(\eta \to \Pi^0 e^+ e^-) = 0.48\text{x}10^{-8}$$

$$\frac{128\Pi^3}{Mg^2} \Gamma(\eta \to \Pi^0 \mu^+ \mu^-) = 0.23\text{x}10^{-7}$$

$$R_1 = 0.7\text{x}10^{-6} \quad , \quad R_2 = 0.4 \times 10^{-5}$$

These numbers should be contrasted with the values from the $\eta \Pi^0 F_{\mu\nu} F_{\mu\nu}$ coupling[3]: $R_1 = 0.9\text{x}10^{-8}$, $R_2 = 1.0\text{x}10^{-5}$. Cheng[4] found $R_1 \cong 10^{-6}$ so our result checks with his. Surprisingly the rate for decay into muons is still larger than that for the decay into electrons but now by a factor of 10 rather than a factor of 10^3. This result is a consequence of the form of the coupling because the terms in A are not appreciably smaller than the terms in B. If we take the rough upper limit set by the recent experiments[6] $\Gamma(\eta \to \Pi^0 2\gamma)/\Gamma(\eta \to \text{all}) \cong 0.1$, then $\Gamma(\eta \to \Pi^0 e^+ e^-)/\Gamma(\eta \to \text{all}) \cong 10^{-7}$ and $\Gamma(\eta \to \Pi^0 \mu^+ \mu^-)/\Gamma(\eta \to \text{all}) \cong 10^{-6}$. The branching ratios are therefore extremely small. Such C-conserving decay modes are not likely to be observed in the near future.

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