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Is There a Quantization Condition for the Classical Problem of Charge and Pole?

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In elementary derivations of the quantization of azimuthal angular momentum the eigenfunction is determined to be $\exp(im\phi)$, *which is "oversensitive" to the rotation* $\phi \to \phi + 2\pi$, *unless* m *is an integer. In a recent paper Kerner examined the classical system of charge and magnetic pole, and expressed* Π *a vector constant of motion for the system, in terms of a physical angle* ψ *to deduce a remarkable paradox. Kerner pointed out that* Π *is "oversensitive" to* $\psi \to \psi + 2\pi$ *unless a certain charge quantization condition is met. Our explicandum of this paradox highlights the distinction between coordinates in classical and quantum physics. It is shown why the single-valuedness requirement on* $\Pi(\psi)$ *is devoid of physical significance. We are finally led to examine the classical analog of the quantum mechanical argument that demonstrates the quantization of magnetic charge, to show that there is "no hope" of a classical quantization condition.*

1. INTRODUCTION

In a recent paper Kerner[1] has derived a remarkable result [Eq. (2) below] for the classical dynamical system comprising a point electron, of charge $-e$ and mass m, in the spherically symmetric field $\mathbf{B} = p\mathbf{x}/r$ of a static monopole

of magnetic charge *p*. It is well known that for this system the ordinary angular momentum **1 = x X** m**v**, where **v** is the electron velocity, is not a conserved vector, although the magnitude of **1,** *ms* = |**1**|, and also v=|**v**| are constants of motion. However, the Poincare angular momentum[2]

$$\mathbf{L} = \mathbf{x} \times m\mathbf{v} + \mu \mathbf{x}/r \qquad (1)$$

where $\mu = ep$, is a constant of motion.'

The formula derived by Kerner[1] is

$$|\mathbf{L}| = (\nu+1)|\mathbf{1}| \qquad (2)$$

where ν is an integer or zero. If this formula of classical mechanics were valid, it would be of special theoretical interest. Dirac,[4] Schwinger,[5] and many others[6.7]s[2] have shown that in quantum mechanics (QM) considerations of rotational invariance, to which some writers, notably Dirac,[4] added gauge invariance requirements, lead to the determination that p is quantized. The significance of this QM result is threefold: (a) The quantization of μ. provides a paradigm example of an (almost) unique and *predicted* coupling constant. (b) The possible magnitude of the magnetic charge of monopoles is predicted. (c) The existence of monopoles would explain the observed quantization of electric charge.

With regard to the QM problem, Hurst,[6] Fierz,[7] and others have emphasized the significance of the Poincare angular momentum **L** defined as[3]

$$\mathbf{L} = \mathbf{x} \times (\mathbf{p} + e\mathbf{A}) + (\mu x/r) \qquad (3)$$

where **A** is chosen so that, except along a string singularity, curl A = p**x**/r[3]. The components of **L** commute with the Hamiltonian of this problem, and also satisfy the commutation rules of the *SO*(3*)* Lie algebra. Rotational covariance requires that the components of **L** be the generators of the *SO*(3) Lie group, so that one finds for quantum states[4]

$$\text{QM:} \qquad |\mu| = \text{integer/half—integer} \qquad (4a)$$

$$\mathbf{L}^2 = l(l+1), \quad l = |\mu|+\nu \qquad (4b)$$

[1] The term involving μ can be identified as the angular momentum of the electric and magnetic fields. Sec Ref. 8.

[2] An extensive bibliography is given in Ref. 3.

[3] Equation (3) is consistent with Eq. **(1)** in classical mechanics since m**v** = **p** + e**A**, where **p** is the dynamical variable canonically conjugate to the position coordinate **x**. In Ref. 1 Kerner also determines potential-free conjugate variables, a problem that is irrelevant in the concerns of this note.

[4] There is as controversy ns In the precise specification of the QM quantization condition. Compare Ref. 4 with Ref. 5 See also Ref. 6.

Kerner's classical formula (2) may be expressed in a form more suitable **for** comparison as

$$\text{Classical (Kerner):} \quad |\mu| = \{\nu(\nu+2)/(\nu+1)^2\}^{1/2}|\mathbf{L}| \qquad (5)$$

In both Eq. (4) and Eq. (5), ν is an integer or zero. Equation (5) certainly looks like a possible classical counterpart of Equation (4).

2. KERNER'S PARADOX

The argument given by Kerner[1].[5] to derive Eq. (2) involves the determination of a classical vector

$$\mathbf{\Pi} = R\mathbf{x} + V\mathbf{v} + S\mathbf{x} \times \mathbf{v} \qquad (6)$$

which is required to be a constant of the motion of an electron in the monopole field. On differentiating (6) and using the equation of motion, one may write down three differential equations([8]) for the unknown scalar functions *R, V,* and S.[6] A suitable parameter to label points on the orbit is given by ϕ, where

$$\sin\phi = \mathbf{x} \cdot \mathbf{v}/rv \qquad (7)$$

In terms of ϕ the general solution for the coefficient of **v** in **Π** is[7]

$$V(\phi) = A[(\cos\Lambda\phi)/\cos\phi] \; B[(\sin\Lambda\phi)/\cos\phi] \qquad (8)$$

where $\Lambda = (1+\mu^2/m^2s^2)^{1/2}$. The key point in the argument given in Ref. 1 is now reached: "... **11** is sensitive to an alteration of [ϕ] by a multiple of 2π, i.e., to physically equivalent representations of the relative orientations of [**x**] and **v.** This oversensitivity is removed when [Λ] is an integer." Thus Kerner deduces Eq. (2).

However, Kerner's argument overlooks the physical meaning of the transformation $\phi \rightarrow \phi + \Delta$ which specifies a motion along the trajectory. Kerner's argument would be meaningful if there were two (possibly equivalent) points ϕ_1 , ϕ_2, on the

[5] For expository reasons our notation differs from that in Ref. 1 in some trivial ways. Note that we use the angle ϕ rather than $\psi = \pi/2$ used by Kerner, so that Eq. (II) is simplified.

[6] There are certain algebraic errors in Eq. (7) of Ref. **1**; these are corrected in Ref. 8.

[7] We note that $\Lambda = 1/\sin\alpha$ where α is the half-angle of the cone on which the trajectory lies.

trajectory for which $\phi_1 \to \phi_2 = 2\pi$. However, the range of ϕ on a trajectory is just π. This is easily proven using the Poincare formula[2]

$$\mathbf{x}^2 = \mathbf{v}^2 t^2 + \mathbf{d}^2$$

in which t is the time, which is zero at the point of closest approach when $\mathbf{x} = \mathbf{d}$,[8] with $|\mathbf{d}| = s/v$. On differentiation of Eq. (9) one obtains

$$\sin \phi = vt/(v^2t^2 + d^2)^{1/2} \tag{10}$$

For completeness we briefly[8] explain the nature of the vector **II** by expressing it in terms of the constant vector **d.** For definiteness we now specify that ϕ lies in the range $-\pi/2$ to $\pi/2$, so that the angle $\Lambda\phi$ appearing in Eq. (9) is the angle between the plane containing L and **d** and the plane containing L and **x.** One can show that[8]

$$\mathbf{\Pi} = A'\mathbf{d} + B'\mathrm{L}\mathbf{x}\mathbf{d} + \mathrm{C}\,\mathbf{L} \tag{1 1}$$

where in terms of the coefficients in Eq. (9), $A' = Av^2/s$ and $B' = Bv^2\Lambda/s|\mathbf{L}|$ while C is arbitrary. With reference to Kerner's argument, we note that if one replaced 0 by 0 + 27r, the effect is to alter the magnitude of A' and B'.

3. THE "NO HOPE" THEOREM

The following argument brings out the essential difference between the QM and the classical description of the electron-monopole system, and explains why there is "no hope" of a quantization condition akin to Eq. (2) in the classical problem. Our basic assumption is that a quantization condition for classical mechanics could only arise by an argument that is parallel to the QM proof of the quantization of We identify the key feature in the various QM arguments as the requirement of rotational covariance and shall show that the peculiar difficulties that arise in QM in fulfilling this requirement are absent in the classical theory.

In QM rotational covariance requires the existence of the operator $U = \exp(\mathrm{iL}\cdot\mathbf{n}\theta)$ for all unit vectors **n** and real angles 0. More formally, one requires the integrability of the *SO(3)* Lie algebra satisfied by the components of **L**, so that there is a set of analytic vectors for L that is dense in the Hilbert space in which state vectors lie.[9] But due to the presence of the

[8] The vector d is analogous to the classic Runge—Lenz vector, which also points from the force center to the closest point of the orbit.

[9] In the detailed discussion given by Hurst in Ref, 6 it is shown that for the monopole potential used by Dirac, the l ,ie algebra of II, is integrable if and only if, 12_1. is an integer.

singular vector potential **A** in **L**, the exponent of U is an unbounded operator, and it follows that in general the transformation U does not exist, and some special requirement must be met if the algebra is to be integrable.[9]

Let us now turn to the classical case. Here, if L is taken as the generator of rotations, and $\boldsymbol{0}$ is any dynamical variable, then under a rotation by θ about the direction

$$\boldsymbol{0}(\mathbf{x}, \mathbf{p}) \to \exp\{\theta n_j [\tfrac{\partial L_j}{\partial p_i}\tfrac{\partial}{\partial x_i} \tfrac{\partial L_j}{\partial x_i}\tfrac{\partial}{\partial p_i}]\}\ \boldsymbol{0}(\mathbf{x}, \mathbf{p}) \tag{12}$$

Expressing **L** in terms of **p** and **x,** one sees that just as in the quantum case, the exponent involves the singular potential **A,** and appears to be highly singular. But the exponential operator is to be applied only to dynamical variables defined on the trajectory, on which $\mathbf{p} + \mathbf{eA} = \mathbf{mv}$, so the apparent singularity is totally innocuous in the classical case.

4. COMMENTS AND DISCUSSION

To a contemporary physicist the argument used by Kerner[1] in the derivation of his classical quantization rule for the charge-pole system is familiar. It closely resembles an elementary derivation of the spectrum of azimuthal angular momentum, such as is to be found in many texts of quantum theory. Such a derivation is the following.

> In the coordinate picture of quantum mechanics the momentum operator is $\mathbf{p} = -i\hbar\nabla$, and the position operator is simply **x,** so that the z component of the angular momentum $\mathbf{l} = \mathbf{x}\times\mathbf{p}$ is $l_z = -i\hbar\partial/\partial\phi$, where ϕ is the azimuthal angle measured about the z axis. The eigenfunctions of l_z are $\exp(im\phi)$. But ϕ and $\phi + 2\pi$ refer to the same point, so that if the wave function is to be single valued, the only permitted values for m are integer or zero, i.e., the eigenvalues of 1_z are $m\hbar$, where m is a (positive or negative) integer (or zero).

This quantum mechanical argument as presented contains the bare bones of more polished expositions. What one can say in distinguishing this discussion from the derivation of Kerner is that the wave function is a field, whereas the classical functions are defined on a line—the trajectory—so that the significance of a coordinate increment such as $\phi \to \phi + \Delta$ is different in the two cases. This point comes out rather more clearly in the proof of'

[10] This equation can be derived by integrating the conventional Poisson bracket expression for an infinitestimal increment.

our "no-hope" theorem, where it is the restriction of classical coordinates to a trajectory that enforces the conclusion.

In Ref. 1 Kerner termed the derived quantization rule "unaccountable," which appears to be an imputation of the conclusion and implicitly of the physical meaningfulness (and soundness) of the derivation. Thus his derivation may properly be termed a paradox, an extraordinary dynamical argument, whose elucidation adds to our comprehension of the distinctions between classical and quantum physics.

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