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Five Classes of Transformations of Dirac Spinors

——The free-particle Dirac equation is brought to "p_0-, "p_1-, "p_2-, "p_3- and "m-linear" forms——

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The free-particle Dirac equation has two remarkable features: (1) It is linear in all four components of the energy-momentum p_μ, and also in the mass m. (2) For its solutions there are five distinct simple modes of the invariant scalar product in the momentum representation.

In this paper, a theorem presented by Case is generalized and used to obtain five classes of transformations of the Dirac equation. Every transformation in a given class has two properties characteristic of the class: (1) The linearity in a corresponding one of the five quantities p_μ, m is maintained in the transformed equation. (In this way "p_0-, "p_1-, "p_2-, "p_3- and "m-linear" forms of the Dirac equation are obtained.) (2) A corresponding mode of the invariant scalar product is preserved. Thus all five classes consist of canonical transformations.

Included amongst the "p_0-linear" forms are the Foldy-Wouthuysen-Tani equation, and the one commonly attributed to Cini and Touschek, together with equations appropriate to limiting situations other than the non-relativistic and extreme relativistic ones. The "canonical" form proposed by Chakrabarti is of the "m-linear" type. Belonging to all three of the "p_1-, "p_2- and "p_3-linear" categories is a "$\boldsymbol{p}$-linear" form of significance for large $|\boldsymbol{p}|$.

§ 1. Introduction

Consider the free-particle Dirac equation for the four-component spinor function $\psi^{(D)}(x)$,

$$(\gamma_\mu p^\mu - m)\psi^{(D)} = 0 , \tag{1·1}$$

where

$$p_\mu = i\partial/\partial x^\mu, \qquad \mu = 0, 1, 2, 3; \tag{1·2}$$

and γ_μ are a set of 4×4 matrices satisfying

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu} . \tag{1·3}$$

(We choose the diagonal metric with $g_{00} = -g_{11} = -g_{22} = -g_{33} = 1$; and we take γ_0, $i\gamma_1$, $i\gamma_2$, $i\gamma_3$ and $i\gamma_5$ $(=i\gamma_0\gamma_1\gamma_2\gamma_3)$ to be hermitean.) A primary feature of (1·1) is the similar (linear) footing given to all four components of the energy-momentum operator p_μ.

This equation was transformed to so-called "canonical" form by Foldy and Wouthuysen,[1] and also independently by Tani.[2] They showed that the wave function

$$\psi^{(F)}(x)=F(\boldsymbol{p},\ m)\psi^{(D)}(x) \tag{1·4}$$

satisfies

$$p_0\psi^{(F)}=E(\boldsymbol{p},\ m)\gamma_0\psi^{(F)}, \tag{1·5}$$

where

$$F(\boldsymbol{p},\ m)=[2E(\boldsymbol{p},\ m)[m+E(\boldsymbol{p},\ m)]]^{-1/2}(m+E(\boldsymbol{p},\ m)+\boldsymbol{\gamma}\cdot\boldsymbol{p}), \tag{1·6}$$

and

$$E(\boldsymbol{p},\ m)=(m^2+\boldsymbol{p}^2)^{1/2}. \tag{1·7}$$

Foldy and Wouthuysen adopted the viewpoint that this procedure is a transformation of the Dirac equation in Hamiltonian form

$$p_0\psi^{(D)}=H\psi^{(D)}, \tag{1·1′}$$

where the Dirac Hamiltonian is

$$H=\gamma_0(\boldsymbol{\gamma}\cdot\boldsymbol{p}+m), \tag{1·8}$$

to yield (1·5) in which the Hamiltonian

$$H^{(F)}=F(\boldsymbol{p},\ m)HF^{-1}(\boldsymbol{p},\ m)=E(\boldsymbol{p},\ m)\gamma_0 \tag{1·9}$$

can be taken to be in diagonal form. [The inverse of the transformation operator is given by

$$F^{-1}(\boldsymbol{p},\ m)=F(-\boldsymbol{p},\ m).] \tag{1·10}$$

The form (1·5) has particular significance when one is considering the non-relativistic limit, $|\boldsymbol{p}|\to 0$.

In contrast to (1·1), (1·5) ascribes a distinctive role to p_0. There is another well-known form of the Dirac equation in which p_0 again appears quite differently from p_i, $i=1, 2, 3$. This form was first put forward by Mendlowitz,[3] and was later rediscovered by Cini and Touschek,[4] and also by Bose, Gamba and Sudarshan.[5] They showed that

$$\psi^{(M)}(x)=M(\boldsymbol{p},\ m)\psi^{(D)}(x) \tag{1·11}$$

satisfies

$$p_0\psi^{(M)}=E(\boldsymbol{p},\ m)|\boldsymbol{p}|^{-1}\gamma_0\boldsymbol{\gamma}\cdot\boldsymbol{p}\psi^{(M)}, \tag{1·12}$$

where

$$M(\boldsymbol{p},\ m)=[2E(\boldsymbol{p},\ m)[|\boldsymbol{p}|+E(\boldsymbol{p},\ m)]]^{-1/2}(|\boldsymbol{p}|+E(\boldsymbol{p},\ m)-|\boldsymbol{p}|^{-1}m\boldsymbol{\gamma}\cdot\boldsymbol{p}). \tag{1·13}$$

In this case

$$M(\boldsymbol{p},\ m)HM^{-1}(\boldsymbol{p},\ m)=E(\boldsymbol{p},\ m)|\boldsymbol{p}|^{-1}\gamma_0\boldsymbol{\gamma}\cdot\boldsymbol{p}, \tag{1·14}$$

where

$$M^{-1}(\boldsymbol{p}, m) = M(-\boldsymbol{p}, m). \tag{1·15}$$

Equation (1·12) is often referred to as "the extreme relativistic form of the Dirac equation", because it is useful when considering the limit $|\boldsymbol{p}|\rightarrow\infty$.

Apart from the Foldy-Wouthuysen-Tani and the Mendlowitz forms, which have been discussed by many authors,[6] there is the "canonical" form as proposed by Chakrabarti.[7] This is

$$\epsilon(p_0)(p_\mu p^\mu)^{1/2}\gamma_0\psi^{(C)} = m\psi^{(C)}, \tag{1·16}$$

with

$$\psi^{(C)}(x) = C(p)\psi^{(D)}(x), \tag{1·17}$$

where

$$C(p) = [2(p_\mu p^\mu)^{1/2}[|p_0| + (p_\nu p^\nu)^{1/2}]]^{-1/2}(|p_0| + (p_\sigma p^\sigma)^{1/2} - \epsilon(p_0)\gamma_0\boldsymbol{\gamma}\cdot\boldsymbol{p}), \tag{1·18}$$

and

$$\epsilon(p_0) = |p_0|^{-1}p_0. \tag{1·19}$$

To be more explicit,

$$C(p)\gamma_\mu p^\mu C^{-1}(p) = \epsilon(p_0)(p_\nu p^\nu)^{1/2}\gamma_0, \tag{1·20}$$

where

$$C^{-1}(p_0, \boldsymbol{p}) = C(p_0, -\boldsymbol{p}). \tag{1·21}$$

Reflecting on these results we are struck by the similarity in form of the transformation operators (1·6), (1·13) and (1·18), suggesting an underlying structure which has not been fully explored. While in (1·1) m and all four components of p_μ appear linearly, we note that this is true only of p_0 in (1·5) and (1·12), and only of m in (1·16). Consequently, labelling (1·5) and (1·12) as "p_0-linear" forms, and (1·16) as an "m-linear" form, of the Dirac equation, we are led to ask if there are other transformations, of a similar structure to those already presented, leading to other "p_0- or "m-linear" equations. Furthermore, can one obtain equations in which any one of the p_i, say p_3, appears linearly; that is, are there "p_3-linear" forms? More importantly, if to either question the answer is yes, are all such transformations canonical?

Another question of more direct physical significance also arises. As has been mentioned, the "p_0-linear" forms (1·5) and (1·12) are in some senses appropriate to the non-relativistic and extreme relativistic situations, respectively. We may then ask, assuming that canonical transformations leading to other "p_0-linear" equations can be obtained, if there are equations amongst these appropriate to other limiting situations; such as, for example, when only p_0 and p_3 become very large (as in a linear accelerator).

Our aim in this paper is to answer these questions. In §2 we state a theorem, which is a generalization of one given by Case.[8] Then in what follows, we exploit the provision by this result of a method of transforming (1·1) into a great variety of forms. In particular, in §§ 3, 4 and 5 respectively, we obtain "p_0-linear" forms, of which (1·5) and (1·12) then appear as special cases; "m-linear" forms, of which (1·16) is a special case; and indeed "p_3-linear" forms. Amongst the "p_0-linear" forms are some appropriate to physical situations in different limits. In § 6, we obtain a "$\boldsymbol{p}$-linear" form which, like the Mendlowitz equation, is most appropriate to the extreme relativistic limit. Finally, in § 7, we go over into momentum representation to discuss the question of scalar products in order to establish that the transformations which yield "p_0-, "p_i- and "m-linear" forms are canonical. We show that, for each of these five types of equations there is associated a mode of scalar product, invariant under inhomogeneous Lorentz transformations, and with respect to which the solutions of the corresponding equation may be taken to form a Hilbert space. The Dirac equation itself is at once of all five types, and accordingly, the scalar product for Dirac wave functions in momentum representation can be expressed in these five modes. The scalar products corresponding to "p_0- and "m-linear" forms have simple expressions also in co-ordinate representation; but this is not the case for those corresponding to "p_i-linear" forms.

In a subsequent publication we hope to exhibit all the results obtained here as manifestations of the group properties of the Dirac equation.

§ 2. Statement of transformation theorem

The main results of this paper are derived using the following theorem, whose proof, being elementary, is not presented:

Theorem: "Let A, B, be $n\times n$ matrices, with

$$\text{(a)}\quad A^2=\alpha^2 I,\qquad B^2=\beta^2 I, \tag{2·1}$$

where I is the $n\times n$ unit matrix, and α, β are real non-zero scalars;

$$\text{(b)}\quad \{B, A-B\}=0\,. \tag{2·2}$$

Define

$$V(A,\alpha;B,\beta)=\beta^{-1}[2\alpha(\alpha+\beta)]^{-1/2}(\beta\alpha I+BA). \tag{2·3}$$

Then

$$\text{(1)}\quad V^{-1}(A,\alpha;B,\beta)=\beta^{-1}[2\alpha(\alpha+\beta)]^{-1/2}(\alpha\beta I+AB) \tag{2·4}$$

$$(=V(B,\alpha;A,\beta)); \tag{2·4'}$$

$$\text{(2)}\quad V(A,\alpha;B,\beta)\;A\;V^{-1}(A,\alpha;B,\beta)=\alpha\beta^{-1}B\,.\text{"} \tag{2·5}$$

Note: 1. If A and B are hermitean matrices, then $V(A,\alpha;B,\beta)$ is a unitary matrix, i.e.

$$V^{-1}(A, \alpha; B, \beta) = V^{\dagger}(A, \alpha; B, \beta). \tag{2·6}$$

(This can only occur with $\alpha^2 > \beta^2$.)

2. Because

$$V(A, \alpha; B, \beta) = -V(A, -\alpha; B, -\beta), \tag{2·7}$$

we shall henceforth take $\alpha > 0$ without any significant loss of generality.

3. If $\alpha^2 = \beta^2$, the choice $\beta > 0$ must also be made to ensure that $[2\alpha(\alpha+\beta)]^{-1/2}$ is well defined. Furthermore, if $\beta^2 > \alpha^2$, $\beta < 0$, then this factor should be replaced by $\pm i[-2\alpha(\alpha+\beta)]^{-1/2}$, with the same choice of sign in (2·3) and (2·4).

4. If $\alpha^2 > \beta^2$, one also has,

(a) with $\beta > 0$,

$$V(A, \alpha; B, \beta) = \exp\left\{\frac{B(A-B)}{2\beta(\alpha^2-\beta^2)^{1/2}} \arctan[(\alpha^2-\beta^2)^{1/2}/\beta]\right\}; \tag{2·8}$$

(b) with $\beta < 0$,

$$V(A, \alpha; B, \beta) = \exp\left\{\frac{B(A-B)}{2\beta(\alpha^2-\beta^2)^{1/2}} (\pi + \arctan[(\alpha^2-\beta^2)^{1/2}/\beta])\right\}. \tag{2·9}$$

5. If $\beta^2 > \alpha^2$, $\beta > 0$, one also has

$$V(A, \alpha; B, \beta) = \exp\left\{\frac{B(A-B)}{2\beta(\beta^2-\alpha^2)^{1/2}} \operatorname{arctanh}[(\beta^2-\alpha^2)^{1/2}/\beta]\right\}. \tag{2·10}$$

In this paper we shall have no need to consider the case $\beta^2 > \alpha^2$, $\beta < 0$.

6. An extension may be made to the situation where α and β are commuting, hermitean operators, whose inverses are defined almost everywhere, and which also commute with both A and B. One is then faced with the difficulty of correctly interpreting the inverse square root of an operator expression, as in (2·3) and (2·4).

7. If an equation holds of the form

$$C\varphi = A\varphi, \tag{2·11}$$

where φ is a vector in the space in which A and C act, and C commutes with $V(A, \alpha; B, \beta)$, then one obtains the equation

$$C\varphi' = \alpha\beta^{-1}B\varphi' \tag{2·12}$$

by defining

$$\varphi' = V(A, \alpha; B, \beta)\varphi. \tag{2·13}$$

§ 3. "p_0-linear" forms of the Dirac equation

Consider now the Dirac Hamiltonian H of (1·8) as the operator A in the terms of § 2, taking

$$\alpha = E(\boldsymbol{p}, m). \tag{3·1}$$

Equation (1·1′) is seen to be of the form (2·11), with $C=p_0$. Thus having found appropriate B and β satisfying the conditions of the theorem, one obtains, in the manner of (2·11)~(2·13), the "p_0-linear" equation

$$p_0\psi' = E(\boldsymbol{p}, m)\beta^{-1}B\psi', \tag{3·2}$$

with

$$\psi'(x) = V(A, \alpha; B, \beta)\psi^{(D)}(x). \tag{3·3}$$

Examples are provided by the following:—

(a) *Foldy-Wouthuysen-Tani form*

Take

$$B=\gamma_0 m\,, \qquad \beta=m\,, \tag{3·4}$$

yielding

$$p_0\psi' = E(\boldsymbol{p}, m)\gamma_0\psi'. \tag{3·5}$$

In fact, in this case

$$V(A, \alpha; B, \beta) = F(\boldsymbol{p}, m). \tag{3·6}$$

(The choice $\beta=-m$ is also possible, leading to a change of sign in (3·5), as seen from (3·2).)

(b) *Mendlowitz form*

Take

$$B=\gamma_0\boldsymbol{\gamma}\cdot\boldsymbol{p}\,, \qquad \beta=|\boldsymbol{p}| \tag{3·7}$$

(noting that singular behaviour may be expected in this case as $|\boldsymbol{p}|\to 0$), yielding

$$p_0\psi' = E(\boldsymbol{p}, m)|\boldsymbol{p}|^{-1}\gamma_0\boldsymbol{\gamma}\cdot\boldsymbol{p}\psi', \tag{3·8}$$

with, in this instance,

$$V(A, \alpha; B, \beta) = M(\boldsymbol{p}, m). \tag{3·9}$$

(Again the replacement of β by $-\beta$ throughout is possible, with a change of sign in (3·8).)

(c) *General "p_0-linear" form*

Take

$$B=\gamma_0(\gamma_1 q_{(1)}+\gamma_2 q_{(2)}+\gamma_3 q_{(3)}+q_{(4)}), \qquad \beta=\pm\{\sum_{a=1}^{4}[q_{(a)}]^2\}^{1/2}, \tag{3·10}$$

where $q_{(a)}(\boldsymbol{p}, m)$ $(a=1, 2, 3, 4)$ are hermitean, and satisfy

$$q_{(1)}[p_1-q_{(1)}]+q_{(2)}[p_2-q_{(2)}]+q_{(3)}[p_3-q_{(3)}]+q_{(4)}[m-q_{(4)}]=0\,. \tag{3·11}$$

(The latter condition is necessary to ensure that (2·2) holds; the former is imposed to ensure the hermiticity of β. Singular behaviour may be expected if $\beta\to 0$ or $\beta=0$ is possible.) In this case, Eq. (3·2) reads

$$p_0\psi' = \pm E(\boldsymbol{p}, m)\{\sum_{\alpha=1}^{4}[q_{(\alpha)}]^2\}^{-1/2}\gamma_0(\gamma_1 q_{(1)} + \gamma_2 q_{(2)} + \gamma_3 q_{(3)} + q_{(4)})\psi'. \quad (3\cdot 12)$$

It is possible to deduce from (3·11) and the hermiticity of $q_{(\alpha)}$ ($a=1, 2, 3, 4$), that $(\alpha^2-\beta^2)$ is positive definite (or zero in the trivial case $B=A$); and this is of course also true of α and β^2. Thus we can in general express $V(A, \alpha; B, \beta)$ in the form (2·8), or (2·9), according as the plus or minus sign is taken in the definition of β in (3·10).

The Foldy-Wouthuysen-Tani and Mendlowitz cases now correspond to the particular choices (with β positive definite in each case)

$$q_{(1)} = q_{(2)} = q_{(3)} = 0\,, \qquad q_{(4)} = m\,; \quad (3\cdot 13)$$

and

$$q_{(1)} = p_1\,, \qquad q_{(2)} = p_2\,, \qquad q_{(3)} = p_3\,, \qquad q_{(4)} = 0\,, \quad (3\cdot 14)$$

respectively.

(d) *Further simple cases*

Other examples of some interest are provided by the following choices in (3·10) (again with β positive definite in each case):

$$\text{(i)} \quad q_{(1)} = p_1\,, \qquad q_{(2)} = 0\,, \qquad q_{(3)} = 0\,, \qquad q_{(4)} = m\,, \quad (3\cdot 15)$$

leading to

$$p_0\psi' = E(\boldsymbol{p}, m)[(p_1)^2 + m^2]^{-1/2}\gamma_0(\gamma_1 p_1 + m)\psi'. \quad (3\cdot 16)$$

(Useful when considering $|p_{2,3}| \to 0$.)

$$\text{(ii)} \quad q_{(1)} = 0\,, \qquad q_{(2)} = p_2\,, \qquad q_{(3)} = p_3\,, \qquad q_{(4)} = 0\,, \quad (3\cdot 17)$$

leading to

$$p_0\psi' = E(\boldsymbol{p}, m)[(p_2)^2 + (p_3)^2]^{-1/2}\gamma_0(\gamma_2 p_2 + \gamma_3 p_3)\psi'. \quad (3\cdot 18)$$

(Useful when considering $|p_{2,3}| \to \infty$.)

$$\text{(iii)} \quad q_{(1)} = 0\,, \qquad q_{(2)} = 0\,, \qquad q_{(3)} = p_3\,, \qquad q_{(4)} = 0\,, \quad (3\cdot 19)$$

leading to

$$p_0\psi' = E(\boldsymbol{p}, m)|p_3|^{-1}\gamma_0\gamma_3 p_3\psi'. \quad (3\cdot 20)$$

(Useful when considering $|p_3| \to \infty$.)

$$\text{(iv)} \quad q_{(i)} = -K(\boldsymbol{p}, m)p_i \;\; (i=1, 2, 3), \quad q_{(4)} = K(\boldsymbol{p}, m)m\,, \quad (3\cdot 21)$$

where

$$K(\boldsymbol{p}, m) = (m^2 - \boldsymbol{p}^2)E^{-2}(\boldsymbol{p}, m), \quad (3\cdot 22)$$

leading to [with $\beta = K(\boldsymbol{p}, m)E(\boldsymbol{p}, m)$]

$$p_0\psi' = \gamma_0(-\boldsymbol{\gamma}\cdot\boldsymbol{p} + m)\psi'. \quad (3\cdot 23)$$

This equation is of interest in regard to the parity transformation for (1·1′). One has

$$V(A, \alpha; B, \beta)=E^{-1}(\boldsymbol{p}, m)(m+\boldsymbol{\gamma}\cdot\boldsymbol{p}), \tag{3·24}$$

so that

$$\psi'(x)[=V(A, \alpha; B, \beta)\psi^{(D)}(x)]=\epsilon(p_0)\gamma_0\psi^{(D)}(x), \tag{3·25}$$

using (1·1′). The relation to the usual parity transformation is clearly seen from (3·25). Another way of looking at the effect of this transformation is provided by the observation that

$$V(A, \alpha; B, \beta)=F^2(\boldsymbol{p}, m). \tag{3·26}$$

The effect of one $F(\boldsymbol{p}, m)$ on $\psi^{(D)}(x)$ is to produce a function satisfying (3·5). If $F^{-1}(\boldsymbol{p}, m)$ is now applied, one of course returns to $\psi^{(D)}(x)$, satisfying (1·1′); but applying instead $F^{-1}(-\boldsymbol{p}, m)$, one must produce a wave function satisfying (3·23), as (3·5) is unchanged if $\boldsymbol{p}\to-\boldsymbol{p}$. However, from (1·10) one sees that

$$F^{-1}(-\boldsymbol{p}, m)=F(\boldsymbol{p}, m), \tag{1·10′}$$

so that the operator $F^2(\boldsymbol{p}, m)$ transforms $\psi^{(D)}(x)$ into a function satisfying (3·23).

§ 4. "*m*-linear" forms

Now consider the Dirac operator $\gamma_\mu p^\mu$ as the operator A in the terms of § 2, taking

$$\alpha=(p_\mu p^\mu)^{1/2}. \tag{4·1}$$

In this case, Eq. (1·1) takes the form (2·11), with $C=m$. For appropriate B and β satisfying the conditions of the theorem, one then obtains the "*m*-linear" equation

$$m\psi'=(p_\mu p^\mu)^{1/2}\beta^{-1}B\psi', \tag{4·2}$$

with

$$\psi'(x)=V(A, \alpha; B, \beta)\psi^{(D)}(x). \tag{4·3}$$

Examples are provided by:—

(a) *Chakrabarti form*

Take

$$B=\gamma_0 p_0, \qquad \beta=|p_0|, \tag{4·4}$$

yielding

$$m\psi'=\epsilon(p_0)(p_\mu p^\mu)^{1/2}\gamma_0\psi'. \tag{4·5}$$

In this case,

$$V(A, \alpha; B, \beta)=C(p). \tag{4·6}$$

(b) *General "m-linear" form*

Take

$$B=\gamma_0 q_{(0)}-\gamma_1 q_{(1)}-\gamma_2 q_{(2)}-\gamma_3 q_{(3)}, \qquad \beta=\{[q_{(0)}]^2-\sum_{i=1}^{3}[q_{(i)}]^2\}^{1/2} \tag{4·7}$$
$$(=[q_{(\nu)}q^{(\nu)}]^{1/2}),$$

where $q_{(\nu)}(p)$ $(\nu=0, 1, 2, 3)$*) are hermitean, are such that β^2 is positive definite and satisfy

$$q_{(0)}[p_0-q_{(0)}]-q_{(1)}[p_1-q_{(1)}]-q_{(2)}[p_2-q_{(2)}]-q_{(3)}[p_3-q_{(3)}]=0\,. \tag{4·8}$$

In this case, Eq. (4·2) reads

$$m\psi'=(p_\mu p^\mu)^{1/2}[q_{(\nu)}q^{(\nu)}]^{-1/2}(\gamma_0 q_{(0)}-\gamma_1 q_{(1)}-\gamma_2 q_{(2)}-\gamma_3 q_{(3)})\psi'. \tag{4·9}$$

It is possible to deduce from (4·8), the hermiticity of $q_{(\nu)}$ $(\nu=0, 1, 2, 3)$, and the positive definiteness of β^2, that $(\beta^2-\alpha^2)$ is also positive definite (or zero in the trivial case $B=A$), so that, with α and β also being positive definite, one can always write $V(A, \alpha; B, \beta)$ in the form (2·10).

The Chakrabarti form corresponds to the choice

$$q_{(0)}=p_0\,,\qquad q_{(1)}=0\,,\qquad q_{(2)}=0\,,\qquad q_{(3)}=0\,. \tag{4·10}$$

(c) *Further simple cases*

Other particular choices of some interest are

$$\text{(i)}\quad q_{(0)}=p_0\,,\qquad q_{(1)}=p_1\,,\qquad q_{(2)}=0\,,\qquad q_{(3)}=0\,, \tag{4·11}$$

leading to

$$m\psi'=(p_\mu p^\mu)^{1/2}[(p_0)^2-(p_1)^2]^{-1/2}(\gamma_0 p_0-\gamma_1 p_1)\psi'. \tag{4·12}$$

$$\text{(ii)}\quad q_{(0)}=W(p)p_0\,,\qquad q_{(i)}=-W(p)p_i \qquad (i=1, 2, 3), \tag{4·13}$$

where

$$W(p)=[(p_0)^2+\boldsymbol{p}^2](p_\mu p^\mu)^{-1}, \tag{4·14}$$

leading to

$$m\psi'=(\gamma_0 p_0+\boldsymbol{\gamma}\cdot\boldsymbol{p})\psi', \tag{4·15}$$

which again has reference to the parity transformation. One has

$$V(A, \alpha; B, \beta)=(p_\mu p^\mu)^{-1/2}\epsilon(p_0)\gamma_0(\gamma_\nu p^\nu), \tag{4·16}$$

so that (using (1·1)) again, as in (3·25),

$$\psi'(x)=\epsilon(p_0)\gamma_0\psi^{(D)}(x)\,. \tag{4·17}$$

Furthermore,

*) We place brackets around indices which may not be Lorentz indices. The distinction between the two types is well brought out in (4·10) for example, where $q(\nu)$ is clearly not a four-vector operator.

$$V(A, \alpha; B, \beta) = C^2(p), \tag{4·18}$$

a fact whose explanation is quite analogous to that following (3·26).

§ 5. "p_3-linear" forms*)

The Dirac equation can be written in the form

$$p_3\psi^{(D)} = G\psi^{(D)}, \tag{5·1}$$

where

$$G = -\gamma_3(\gamma_0 p_0 - \gamma_1 p_1 - \gamma_2 p_2 - m). \tag{5·2}$$

Taking G now as A in the terms of § 2, with

$$\alpha = [(p_0)^2 - (p_1)^2 - (p_2)^2 - m^2]^{1/2} \quad (= \lambda(p_0, p_1, p_2, m), \text{ say}), \tag{5·3}$$

one can again look for suitable B and β. Equation (5·1) is of the form (2·11) with $C = p_3$; so that when B and β are found, one obtains the "p_3-linear" equation

$$p_3\psi' = \lambda(p_0, p_1, p_2, m)\beta^{-1}B\psi', \tag{5·4}$$

with

$$\psi'(x) = V(A, \alpha; B, \beta)\psi^{(D)}(x). \tag{5·5}$$

(Note that singular behaviour may be expected as $\lambda(p_0, p_1, p_2, m) \to 0$.)

Examples are provided by:—

(a) *General "p_3-linear" form*

Take

$$B = -\gamma_3(\gamma_0 q_{(0)} - \gamma_1 q_{(1)} - \gamma_2 q_{(2)} - q_{(3)}), \qquad \beta = \{[q_{(0)}]^2 - \sum_{i=1}^{3}[q_{(i)}]^2\}^{1/2} \tag{5·6}$$

$$(= [q_{(\nu)}q^{(\nu)}]^{1/2}),$$

where $q_{(\nu)}(p_0, p_1, p_2, m)$ $(\nu = 0, 1, 2, 3)$**) are hermitean, are such that β^2 is positive definite, and satisfy

$$q_{(0)}[p_0 - q_{(0)}] - q_{(1)}[p_1 - q_{(1)}] - q_{(2)}[p_2 - q_{(2)}] - q_{(3)}[m - q_{(3)}] = 0. \tag{5·7}$$

Then (5·4) reads

$$p_3\psi' = -\lambda(p_0, p_1, p_2, m)[q_{(\nu)}q^{(\nu)}]^{-1/2}\gamma_3(\gamma_0 q_{(0)} - \gamma_1 q_{(1)} - \gamma_2 q_{(2)} - q_{(3)})\psi'. \tag{5·8}$$

In analogy with the case of the general "m-linear" form, $V(A, \alpha; B, \beta)$ here can always be written in the form (2·10).

(b) *A simple case*

A choice of $q_{(\nu)}$ $(\nu = 0, 1, 2, 3)$ of some interest is

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*) By obvious alteration of the discussion in this section of "p_3-linear" cases, analogous results for "p_1- and "p_2-linear" cases can be derived.

**) See the footnote on p. 824.

$$q_{(0)}=p_0\,,\qquad q_{(1)}=0\,,\qquad q_{(2)}=0\,,\qquad q_{(3)}=0\,,\tag{5·9}$$

leading to

$$p_3\psi'=\epsilon(p_0)\lambda(p_0,p_1,p_2,m)\gamma_0\gamma_3\psi',\tag{5·10}$$

which is the simplest "p_3-linear" form one can obtain.

§ 6. Further applications of transformation theorem

We apply the theorem in yet another way to obtain from the Dirac equation an interesting "$\boldsymbol{p}$-linear" form:—

Multiply (1·1) by γ_5 to obtain

$$\gamma_5(\gamma_0 p_0-m)\psi^{(D)}=\gamma_5\boldsymbol{\gamma}\cdot\boldsymbol{p}\psi^{(D)}.\tag{6·1}$$

Now apply the theorem with

$$A=\gamma_5(\gamma_0 p_0-m),\qquad \alpha=[(p_0)^2-m^2]^{1/2}\,;\tag{6·2}$$

$$B=\gamma_5\gamma_0 p_0\,,\qquad \beta=|p_0|,\tag{6·3}$$

(and $C=\gamma_5\boldsymbol{\gamma}\cdot\boldsymbol{p}$), to obtain

$$\epsilon(p_0)[(p_0)^2-m^2]^{1/2}\gamma_5\gamma_0\psi'=\gamma_5\boldsymbol{\gamma}\cdot\boldsymbol{p}\psi',\tag{6·4}$$

or, finally,

$$\epsilon(p_0)[(p_0)^2-m^2]^{1/2}\gamma_0\psi'=\boldsymbol{\gamma}\cdot\boldsymbol{p}\psi'.\tag{6·4'}$$

This equation is similar in appearance to the Mendlowitz form (1·12), and is just as useful in considering the limit $|\boldsymbol{p}|\to\infty$. In both cases, states of positive and negative helicity are separately described by two component equations. In (6·4'), the transformed wave function is

$$\psi'(x)=V(A,\alpha;B,\beta)\psi^{(D)}(x),\tag{6·5}$$

where, explicitly,

$$V(A,\alpha;B,\beta)=\{2[(p_0)^2-m^2]^{1/2}(|p_0|+[(p_0)^2-m^2]^{1/2})\}^{-1/2}$$
$$\times(|p_0|+[(p_0)^2-m^2]^{1/2}-\epsilon(p_0)m\gamma_0).\tag{6·6}$$

Note that (6·4') is in particular "p_3-linear", and accordingly an equation of this form could have been obtained via the method of § 5(a).

As regards applications of the theorem other than to the free-particle Dirac equation, we mention that Case [8] has applied his restricted version in dealing with free-particle equations for spin 0 and 1, and with one-particle equations in situations involving electro-magnetic interactions. An example of the theorem's implicit use is contained in a paper by Biedenharn.[9]

§ 7. Scalar products

It may be said that a form of the Dirac equation is of limited value unless one can exhibit the solutions thereof as forming a Hilbert space which carries the appropriate representations of the full inhomogeneous Lorentz group for the description of a free spin $\frac{1}{2}$ particle (and anti-particle) of mass m. Thus it is important to find an invariant scalar product for solutions of a given form of the equation. It is in fact possible to do this for each of the three general forms discussed in §§ 3, 4 and 5.

To consider this question, it is convenient to go over into momentum representation. Thus we write

$$\psi^{(D)}(x_0, \boldsymbol{x}) = (2\pi)^{-3/2} \int \frac{d^3k}{E(\boldsymbol{k}, m)} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \{e^{-iE(\boldsymbol{k},m)x_0}\chi^{(D)+}(\boldsymbol{k}) + e^{+iE(\boldsymbol{k},m)x_0}\chi^{(D)-}(\boldsymbol{k})\}, \tag{7·1}$$

where

$$(\gamma_\mu k^\mu - m)\chi^{(D)\pm} = 0, \tag{7·2}$$

with

$$k_0\chi^{(D)\pm} = \pm E(\boldsymbol{k}, m)\chi^{(D)\pm}. \tag{7·3}$$

Conversely to (7·1), one has

$$\chi^{(D)\pm}(\boldsymbol{k}) = e^{\pm iE(\boldsymbol{k},m)x_0}\tfrac{1}{2}[E(\boldsymbol{k}, m) \pm p_0](2\pi)^{-3/2} \int d^3x e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\psi^{(D)}(x_0, x). \tag{7·4}$$

Now the scalar product for Dirac wave functions in coordinate representation is well known as

$$(\psi_1^{(D)}, \psi_2^{(D)}) = \int_{x_0\ \mathrm{const}} d^3x \psi_1^{(D)\dagger}(x_0, \boldsymbol{x})\psi_2^{(D)}(x_0, \boldsymbol{x}), \tag{7·5}$$

where $\psi^{(D)\dagger}$ is the hermitean conjugate of $\psi^{(D)}$. This scalar product is invariant under inhomogeneous Lorentz transformations. It can also be written in the form[10]

$$(\psi_1^{(D)}, \psi_2^{(D)}) = \frac{i}{2m} \int_{x_0\ \mathrm{const}} d^3x \left\{\overline{\psi}_1^{(D)}(x_0, \boldsymbol{x}) \frac{\partial\psi_2^{(D)}(x_0, \boldsymbol{x})}{\partial x_0} - \frac{\partial\overline{\psi}_1^{(D)}(x_0, \boldsymbol{x})}{\partial x_0}\psi_2^{(D)}(x_0, \boldsymbol{x})\right\}, \tag{7·6}$$

where

$$\overline{\psi}^{(D)} = \psi^{(D)\dagger}\gamma_0. \tag{7·7}$$

Using (7·1) one finds[11]

$$(\psi_1^{(D)}, \psi_2^{(D)}) = \int \frac{d^3k}{E^2(\boldsymbol{k}, m)} \{\chi_1^{(D)+\dagger}(\boldsymbol{k})\chi_2^{(D)+}(\boldsymbol{k}) + \chi_1^{(D)-\dagger}(\boldsymbol{k})\chi_2^{(D)-}(\boldsymbol{k})\}. \tag{7·8}$$

(The cross terms vanish because of the hermiticity of the operator $E^{-1}(\boldsymbol{k}, m)\gamma_0(\boldsymbol{\gamma}\cdot\boldsymbol{k} + m)$, which has different eigenvalues [viz. $+1$, -1 respectively] on $\chi^{(D)+}$ and $\chi^{(D)-}$.)

From (7·2) it is possible to obtain the identity[12]

$$k_\mu \bar{\chi}_1^{(D)\pm}(\boldsymbol{k})\chi_2^{(D)\pm}(\boldsymbol{k}) = m\bar{\chi}_1^{(D)\pm}(\boldsymbol{k})\gamma_\mu\chi_2^{(D)\pm}(\boldsymbol{k}). \tag{7·9}$$

In particular,

$$k_0 \bar{\chi}_1^{(D)\pm}(\boldsymbol{k})\chi_2^{(D)\pm}(\boldsymbol{k}) = m\chi_1^{(D)\pm\dagger}(\boldsymbol{k})\chi_2^{(D)\pm}(\boldsymbol{k}), \tag{7·9′}$$

allowing one to deduce[11] from (7·8),

$$(\psi_1^{(D)}, \psi_2^{(D)}) = \int \frac{d^3k}{mE(\boldsymbol{k}, m)} \{\bar{\chi}_1^{(D)+}(\boldsymbol{k})\chi_2^{(D)+}(\boldsymbol{k}) - \bar{\chi}_1^{(D)-}(\boldsymbol{k})\chi_2^{(D)-}(\boldsymbol{k})\}. \tag{7·10}$$

Furthermore, one can now use

$$k_3 \bar{\chi}_1^{(D)\pm}(\boldsymbol{k})\chi_2^{(D)\pm}(\boldsymbol{k}) = m\bar{\chi}_1^{(D)\pm}(\boldsymbol{k})\gamma_3\chi_2^{(D)\pm}(\boldsymbol{k}), \tag{7·9″}$$

to deduce from (7·10),

$$(\psi_1^{(D)}, \psi_2^{(D)}) = \int \frac{d^3k}{E(\boldsymbol{k}, m)k_3} \{\bar{\chi}_1^{(D)+}(\boldsymbol{k})\gamma_3\chi_2^{(D)+}(\boldsymbol{k}) - \bar{\chi}_1^{(D)-}(\boldsymbol{k})\gamma_3\chi_2^{(D)-}(\boldsymbol{k})\}. \tag{7·11}$$

The three modes (7·8), (7·10) and (7·11) of the scalar product for Dirac wave functions are of particular significance in regard to "p_0-, "m-, and "p_3-linear" forms, respectively, of the Dirac equation.

For the general "p_0-linear" case [§ 3(c)] in momentum representation, one has

$$A = \gamma_0(\boldsymbol{\gamma}\cdot\boldsymbol{k} + m), \qquad \alpha = E(\boldsymbol{k}, m); \tag{7·12}$$

$$B = \gamma_0(\gamma_1 q_{(1)} + \gamma_2 q_{(2)} + \gamma_3 q_{(3)} + q_{(4)}), \qquad \beta = \pm\{\sum_{a=1}^{4}[q_{(a)}]^2\}^{1/2}, \tag{7·13}$$

with $q_{(a)}(\boldsymbol{k}, m)$ $(a=1, 2, 3, 4)$ real. Thus A and B are hermitean matrices, so that $V(A, \alpha; B, \beta)$ is a unitary matrix [cf. (2·6)]. Defining

$$(\psi_1', \psi_2') = \int \frac{d^3k}{E^2(\boldsymbol{k}, m)} \{\chi_1'^{+\dagger}(\boldsymbol{k})\chi_2'^{+}(\boldsymbol{k}) + \chi_1'^{-\dagger}(\boldsymbol{k})\chi_2'^{-}(\boldsymbol{k})\}, \tag{7·14}$$

with

$$\chi'^{\pm}(\boldsymbol{k}) = V(A, \alpha; B, \beta)\chi^{(D)\pm}(\boldsymbol{k}) \tag{7·15}$$

[so that $\chi'^{\pm}(\boldsymbol{k})$ are related to $\psi'(x)$ of (3·12) in the manner of (7·1), (7·4)], one then has, from (7·8),

$$(\psi_1', \psi_2') = (\psi_1^{(D)}, \psi_2^{(D)}). \tag{7·16}$$

For the general "m-linear" case [§ 4(b)], one has

$$A = \gamma_\mu k^\mu, \qquad \alpha = (k_\mu k^\mu)^{1/2}; \tag{7·17}$$

$$B = \gamma_0 q_{(0)} - \gamma_1 q_{(1)} - \gamma_2 q_{(2)} - \gamma_3 q_{(3)}, \qquad \beta = [q_{(\nu)}q^{(\nu)}]^{1/2}, \tag{7·18}$$

with $q_{(\nu)}(k)$ $(\nu=0, 1, 2, 3)$ real. Thus A and B satisfy

$$\gamma_0 A^\dagger = A\gamma_0, \qquad \gamma_0 B^\dagger = B\gamma_0, \tag{7·19}$$

enabling one to deduce that

$$\gamma_0 V^\dagger(A, \alpha; B, \beta) = V^{-1}(A, \alpha; B, \beta)\gamma_0 . \tag{7·20}$$

(Note that $A^\dagger$ for example means here the usual matrix hermitean conjugate of A; there is no reference to a particular mode of Dirac scalar product.) Defining

$$(\psi_1', \psi_2') = \int \frac{d^3k}{mE(\boldsymbol{k}, m)} \{\bar{\chi}_1'^{+}(\boldsymbol{k})\chi_2'^{+}(\boldsymbol{k}) - \bar{\chi}_1'^{-}(\boldsymbol{k})\chi_2'^{-}(\boldsymbol{k})\}, \tag{7·21}$$

with, again,

$$\chi'^{\pm}(\boldsymbol{k}) = V(A, \alpha; B, \beta)\chi^{(D)\pm}(\boldsymbol{k}), \tag{7·22}$$

one has, from (7·20), (7·10),

$$(\psi_1', \psi_2') = (\psi_1^{(D)}, \psi_2^{(D)}). \tag{7·23}$$

Finally, for the general "p_3-linear" case [§ 5(a)], one has

$$A = -\gamma_3(\gamma_0 k_0 - \gamma_1 k_1 - \gamma_2 k_2 - m), \qquad \alpha = \lambda(k_0, k_1, k_2, m); \tag{7·24}$$

$$B = -\gamma_3(\gamma_0 q_{(0)} - \gamma_1 q_{(1)} - \gamma_2 q_{(2)} - q_{(3)}), \qquad \beta = [q_{(\nu)} q^{(\nu)}]^{1/2}, \tag{7·25}$$

with $q_{(\nu)}(k_0, k_1, k_2, m)$ $(\nu = 0, 1, 2, 3)$ real. Then A and B satisfy

$$\gamma_0\gamma_3 A^\dagger = A\gamma_0\gamma_3 , \qquad \gamma_0\gamma_3 B^\dagger = B\gamma_0\gamma_3 , \tag{7·26}$$

whence one deduces

$$\gamma_0\gamma_3 V^\dagger(A, \alpha; B, \beta) = V^{-1}(A, \alpha; B, \beta)\gamma_0\gamma_3 .^{*)} \tag{7·27}$$

Defining

$$(\psi_1', \psi_2') = \int \frac{d^3k}{E(\boldsymbol{k}, m)k_3} \{\bar{\chi}_1'^{+}(\boldsymbol{k})\gamma_3\chi_2'^{+}(\boldsymbol{k}) - \bar{\chi}_1'^{-}(\boldsymbol{k})\gamma_3\chi_2'^{-}(\boldsymbol{k})\}, \tag{7·28}$$

with

$$\chi'^{\pm}(\boldsymbol{k}) = V(A, \alpha; B, \beta)\chi^{(D)\pm}(\boldsymbol{k}), \tag{7·29}$$

one has, from (7·27), (7·11),

$$(\psi_1', \psi_2') = (\psi_1^{(D)}, \psi_2^{(D)}). \tag{7·30}$$

The Dirac wave functions form a Hilbert space with the associated Lorentz-invariant scalar product (7·8) [(7·10), (7·11)]. The results (7·16), (7·23) and (7·30) show that in transforming the Dirac equation into a "p_0-, "m- or "p_3-linear" form**) via the use of an operator $V(A, \alpha; B, \beta)$, one is in effect performing a canonical transformation to a new description of the free spin $\frac{1}{2}$ particle (and anti-particle) of mass m. The solutions of the derived equation form a Hilbert space with associated scalar product (7·14), (7·21) or (7·28)

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*) Note that this is also true for $V(A, \alpha; B, \beta)$ as in (6·6). The following results for "p_3-linear" forms thus also hold for (6·4′).

**) See the first footnote on p. 825.

as the case may be; and this scalar product is invariant under inhomogenous Lorentz transformations as represented in the new picture. Just as the Hilbert space of Dirac wave functions carries the appropriate representations of the full inhomogeneous Lorentz group, so also this is true of the derived space.

Note that the unitarity of the *matrix* $V(A, \alpha; B, \beta)$ in the general "p_0-linear" case is in this context of no more and no less significance than the relationships (7·20) and (7·27); it is in particular misleading to refer to the Foldy-Wouthuysen-Tani transformation, but not the Chakrabarti one, as unitary.

In closing, we point out that the scalar products (7·14) and (7·21) also have simple forms in the co-ordinate representation, viz.

$$(\psi_1', \psi_2') = \int_{x_0\ \mathrm{const}} d^3x \psi_1'^{\dagger}(x_0, \boldsymbol{x}) \psi_2'(x_0, \boldsymbol{x}), \tag{7·31}$$

corresponding to (7·14); and

$$(\psi_1', \psi_2') = \frac{i}{2m} \int_{x_0\ \mathrm{const}} d^3x \left\{ \overline{\psi}_1'(x_0, \boldsymbol{x}) \frac{\partial \psi_2'(x_0, \boldsymbol{x})}{\partial x_0} - \frac{\partial \overline{\psi}_1'(x_0, \boldsymbol{x})}{\partial x_0} \psi_2'(x_0, \boldsymbol{x}) \right\}, \tag{7·32}$$

corresponding to (7·21). There is no simple form in the co-ordinate representation of the scalar products corresponding to "p_i-linear" forms.

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