Hence, we have Eq. (I.16):

$$i\frac{\partial\phi^{(1)}}{\partial\tau}+p\frac{\partial^2\phi^{(1)}}{\partial\xi^2}+q\,|\phi^{(1)}|^2\,\phi^{(1)}=0,$$

with

$$p=m^2/2\omega^3,$$

$$q=-3\kappa/2\omega.$$

For a negative $\kappa$, $pq$ is positive so that this equation admits the solitary-wave solution, such as that given by Eq. (I.17), while the plane waves are modulationally unstable as was stated before in Eq. (I.18). On the other hand, if $\kappa$ is positive, we have the solutions such as those given in Sec. 2 for the plasma wave, and the equation can be reduced to the Kortweg–de Vries equation.

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# On Canonical $SO(4, 1)$ Transformations of the Dirac Equation

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Certain matrix transformations of the free-particle Dirac equation are described as momentum-dependent $SO(4, 1)$ transformations. Such of these belonging to any one of five subgroups $G^{(\alpha)}$ ($\alpha$ = 0, 1, 2, 3, 4) are canonical, preserving the Lorentz-invariant Dirac scalar product in a corresponding one of five modes of expression. The Dirac equation itself is linear in all five components $p_\alpha$ [$p_\mu$ ($\mu$ = 0, 1, 2, 3) is the four-momentum operator, and $p_4 = m$] of the "five-vector" $\tilde{p}$, and a transformation in $G^{(\beta)}$ has the additional property that the component $p_\beta$ appears linearly also in the transformed equation. The Mendlowitz and the Foldy–Wouthuysen–Tani transformation accordingly are in $G^{(0)}$, the $SO(4)$ subgroup; and that proposed by Chakrabarti is in $G^{(4)}$, the $SO(3, 1)$ subgroup associated with homogeneous Lorentz transformations. For any $\tilde{p}'$, obtained from $\tilde{p}$ by a momentum-dependent $SO(4, 1)$ transformation, there is a corresponding transform of the Dirac equation. Where $p_\alpha$ appears in the Dirac equation, $p'_\alpha$ appears in the transformed equation. The ambiguities which arise in the specification of the transformation leading to a given such equation are associated with the existence of a "little group" for any such $\tilde{p}'$.

## 1. INTRODUCTION

The Dirac equation for the four-component wave-function $\psi^{(D)}(x)$ is

$$(\gamma_\mu p^\mu - m)\psi^{(D)}=0, \tag{1.1}$$

where

$$p_\mu=\frac{i\partial}{\partial x^\mu},\quad \mu=0,1,2,3, \tag{1.2}$$

and the matrices $\gamma_\mu$ form an irreducible representation of the Dirac–Clifford algebra, with

$$\{\gamma_\mu,\gamma_\nu\}=2g_{\mu\nu}. \tag{1.3}$$

[We choose the diagonal metric with $g_{00} = -g_{11} = -g_{22} = -g_{33} = 1$; and, with no significant loss of generality, we take $\gamma_0$, $i\gamma_1$, $i\gamma_2$, $i\gamma_3$, and $i\gamma_5$ ($= i\gamma_0\gamma_1\gamma_2\gamma_3$) to be Hermitian.]

The transformation properties of the bispinor function $\psi^{(D)}(x)$ with respect to the restricted homogeneous Lorentz group $SO(3, 1)$ are well known. However, it has also been long known that larger groups, in fact certain groups of rotations in five- and six-dimensional spaces, are pertinent to discussions of the Dirac equation, the Lie algebra of the Dirac matrices $\gamma_\mu$, etc.[1–7] In this paper, the connection between Eq. (1.1) and the group $SO(4, 1)$ in particular is exploited in the development of a unifying group-theoretical description of certain canonical momentum-dependent transformations of the equation.

In a previous publication,[8] henceforth referred to as BC, two features of Eq. (1.1) assume significance: namely, the linearity in all five of the quantities $p_\mu$, $m$, and the existence of five different ways of expressing one and the same Lorentz-invariant scalar product

[1] A. S. Eddington, Proc. Roy. Soc. (London) **121A**, 524 (1928).
[2] P. A. M. Dirac, Ann. Math. **36**, 657 (1935).
[3] Harish-Chandra, Proc. Indian Acad. Sci. **22A**, 30 (1945).
[4] A. O. Barut, Phys. Rev. **135**, B839 (1964). See also A. J. MacFarlane, Commun. Math. Phys. **2**, 133 (1966); A. ten Kate, J. Math. Phys. **9**, 181 (1968).
[5] J. K. Lubański, Physica **9**, 310 (1942). See also H. J. Bhabha, Rev. Mod. Phys. **17**, 200 (1945); J. A. de Vos and J. Hilgevoord, Nucl. Phys. **B1**, 494 (1967); M. M. Bakri, J. Math. Phys. **10**, 298 (1969).
[6] See, for example, C. Fronsdal, Proc. Roy. Soc. (London) **288A**, 113 (1965), and references given therein.
[7] A. O. Barut, Phys. Rev. Letters **20**, 893 (1968).
[8] A. J. Bracken and H. A. Cohen, Progr. Theoret. Phys. **41**, 816 (1969).

$(\psi_1^{(D)}, \psi_2^{(D)})$ of any two solutions $\psi_{1,2}^{(D)}(x)$. The five expressions are all simple in the momentum representation, introduced by defining

$$\psi^{(D)}(x_0, \mathbf{x}) = (2\pi)^{-\frac{3}{2}}\int \frac{d^3k}{\omega(\mathbf{k}, m)} e^{i\mathbf{k}\cdot\mathbf{x}}\{e^{-i\omega(\mathbf{k},m)x_0}\chi^{(D)+}(\mathbf{k}) + e^{+i\omega(\mathbf{k},m)x_0}\chi^{(D)-}(\mathbf{k})\}, \quad (1.4)$$

where

$$\omega(\mathbf{k}, m) = (\mathbf{k}^2 + m^2)^{\frac{1}{2}}, \quad (1.5)$$

and with, conversely,

$$\chi^{(D)\pm}(\mathbf{k}) = e^{\pm i\omega(\mathbf{k},m)x_0}\tfrac{1}{2}[\omega(\mathbf{k}, m) \pm p_0](2\pi)^{-\frac{3}{2}} \times \int d^3x e^{-i\mathbf{k}\cdot\mathbf{x}}\psi^{(D)}(x_0, \mathbf{x}). \quad (1.6)$$

Then the well-known coordinate representation form

$$(\psi_1^{(D)}, \psi_2^{(D)}) = \int_{x_0 \text{ const}} d^3x \psi_1^{(D)\dagger}(x_0, \mathbf{x})\psi_2^{(D)}(x_0, \mathbf{x}), \quad (1.7)$$

where $\psi^{(D)\dagger}$ is the Hermitian conjugate of $\psi^{(D)}$, yields the five expressions

$$(\psi_1^{(D)}, \psi_2^{(D)}) = \int \frac{d^3k}{\omega^2(\mathbf{k}, m)} \{\chi_1^{(D)+\dagger}(\mathbf{k})\chi_2^{(D)+}(\mathbf{k}) + \chi_1^{(D)-\dagger}(\mathbf{k})\chi_2^{(D)-}(\mathbf{k})\} \quad (1.8a)$$

$$= \int \frac{d^3k}{m\omega(\mathbf{k}, m)} \{\bar{\chi}_1^{(D)+}(\mathbf{k})\chi_2^{(D)+}(\mathbf{k}) - \bar{\chi}_1^{(D)-}(\mathbf{k})\chi_2^{(D)-}(\mathbf{k})\} \quad (1.8b)$$

and

$$= \int \frac{d^3k}{\omega(\mathbf{k}, m)k_i} \{\bar{\chi}_1^{(D)+}(\mathbf{k})\gamma_i\chi_2^{(D)+}(\mathbf{k}) - \bar{\chi}_1^{(D)-}(\mathbf{k})\gamma_i\chi_2^{(D)-}(\mathbf{k})\}$$

$$(i = 1, 2, \text{ or } 3; \text{ no summation}); \quad (1.8c)$$

where

$$\bar{\chi}^{(D)\pm} = \chi^{(D)\pm\dagger}\gamma_0. \quad (1.9)$$

The equivalence of these five expressions is established using the identity

$$k_\mu\bar{\chi}_1^{(D)\pm}(\mathbf{k})\chi_2^{(D)\pm}(\mathbf{k}) = m\bar{\chi}_1^{(D)\pm}(\mathbf{k})\gamma_\mu\chi_2^{(D)\pm}(\mathbf{k}), \quad (1.10)$$

where we define

$$k_0\chi^{(D)\pm}(\mathbf{k}) = \pm\omega(\mathbf{k}, m)\chi^{(D)\pm}(\mathbf{k}). \quad (1.11)$$

Equation (1.10) in turn follows[9] from the fact that one has, from (1.1) and (1.6),

$$(\gamma_\mu k^\mu - m)\chi_{1,2}^{(D)\pm} = 0. \quad (1.12)$$

In BC we considered momentum-dependent matrix transformations of the form

$$\psi^{(D)}(x) \rightarrow \psi'(x) = V(p, m)\psi^{(D)}(x), \quad (1.13)$$

leading to the equation

$$V(p, m)[\gamma_\mu p^\mu - m]V^{-1}(p, m)\psi' = 0. \quad (1.14)$$

Five special classes of such transformations were presented, with every transformation in a given class having two properties characteristic of that class.

The first of these properties is that the linearity in a corresponding one of the five quantities $p_\mu$, $m$ is maintained in the transformed equation. In this way "$p_0$-, "$p_1$-, "$p_2$-, "$p_3$-, and "$m$-linear" equations are obtained.

Amongst the "$p_0$-linear" forms, one finds the Foldy–Wouthuysen–Tani[10] equation

$$p_0\psi^{(F)} = \gamma_0\omega(\mathbf{p}, m)\psi^{(F)}, \quad (1.15)$$

with, in this case,

$$\psi^{(F)}(x) = F(\mathbf{p}, m)\psi^{(D)}(x), \quad (1.16)$$

where

$$F(\mathbf{p}, m) = \exp\left[\frac{\boldsymbol{\gamma}\cdot\mathbf{p}}{2\,|\mathbf{p}|}\arctan\left(\frac{|\mathbf{p}|}{m}\right)\right]. \quad (1.17)$$

Also of the "$p_0$-linear" type is the equation proposed by Mendlowitz[11]:

$$p_0\psi^{(M)} = \omega(\mathbf{p}, m)\gamma_0\frac{\boldsymbol{\gamma}\cdot\mathbf{p}}{|\mathbf{p}|}\psi^{(M)}, \quad (1.18)$$

where

$$\psi^{(M)}(x) = M(\mathbf{p}, m)\psi^{(D)}(x), \quad (1.19)$$

with

$$M(\mathbf{p}, m) = \exp\left[-\frac{\boldsymbol{\gamma}\cdot\mathbf{p}}{2\,|\mathbf{p}|}\arctan\left(\frac{m}{|\mathbf{p}|}\right)\right]. \quad (1.20)$$

Amongst the "$m$-linear" equations is that proposed by Chakrabarti[12]:

$$\epsilon(p_0)(p_\mu p^\mu)^{\frac{1}{2}}\gamma_0\psi^{(C)} = m\psi^{(C)}, \quad (1.21)$$

where

$$\epsilon(p_0) = p_0\,|p_0|^{-1} \quad (1.22)$$

and

$$\psi^{(C)}(x) = C(p)\psi^{(D)}(x), \quad (1.23)$$

[9] The proof is a simple extension of that for the case $\chi_1^{(D)\pm} = \chi_2^{(D)\pm}$ as given, for example, in S. S. Schweber, *An Introduction to Relativistic Quantum Field Theory* (Row Peterson & Company, New York, 1961), Chap. 4, preceding equation (129).

[10] L. L. Foldy and S. A. Wouthuysen, Phys. Rev. **78**, 29 (1950); S. Tani, Progr. Theoret. Phys. (Kyoto) **6**, 267 (1951). The transformation was in fact earlier proposed by M. H. L. Pryce, Proc. Roy. Soc. (London) **195A**, 62 (1948).

[11] H. Mendlowitz, Phys. Rev. **102**, 527 (1956). The transformation was rediscovered by M. Cini and B. Touschek, Nuovo Cimento **7**, 422 (1958); and independently by S. K. Bose, A. Gamba, and E. C. G. Sudarshan, Phys. Rev. **113**, 1661 (1959).

[12] A. Chakrabarti, J. Math. Phys. **4**, 1215, 1223 (1963).

with

$$C(p) = \exp\left\{-\frac{\epsilon(p_0)\gamma_0\boldsymbol{\gamma}\cdot\mathbf{p}}{2\,|\mathbf{p}|}\operatorname{arc\,tanh}\left(\frac{|\mathbf{p}|}{|p_0|}\right)\right\}. \quad (1.24)$$

The simplest "$p_3$-linear" form obtained is

$$p_3\psi' = \epsilon(p_0)\lambda(p_0, p_1, p_2, m)\gamma_0\gamma_3\psi', \quad (1.25)$$

where

$$\lambda(p_0, p_1, p_2, m) = [(p_0)^2 - (p_1)^2 - (p_2)^2 - m^2]^{\frac{1}{2}} \quad (1.26)$$

and

$$\psi'(x) = W(p_0, p_1, p_2, m)\psi^{(D)}(x), \quad (1.27)$$

with

$$W(p_0, p_1, p_2, m) = \exp\left\{-\frac{\epsilon(p_0)\gamma_0(\gamma_1 p_1 + \gamma_2 p_2 + m)}{2[(p_1)^2 + (p_2)^2 + m^2]^{\frac{1}{2}}} \times \operatorname{arc\,tanh}\left(\frac{[(p_1)^2 + (p_1)^2 + m^2]^{\frac{1}{2}}}{|p_0|}\right)\right\}. \quad (1.28)$$

The second property characterizing a given one of the five classes is that every transformation within that class preserves the Dirac scalar product in a corresponding one of the five modes (1.8a)–(1.8c), so that each class consists of canonical transformations. In fact, the transformations leading to "$p_0$-, "$m$-, and "$p_i$-linear" forms preserve the modes (1.8a)–(1.8c), respectively.

In describing a subset of transformations (1.13) as momentum-dependent $SO(4, 1)$ transformations, making use of the connection between Eq. (1.1) and this group, we aim here in particular to interpret the above results of BC in group-theoretical terms. To this end, in Sec. 2, we make explicit this connection to the extent required in what follows.

In Sec. 3, the significance of such a connection in regard to transformations of the form (1.13) is established. We stress in particular the existence of five subgroups of $SO(4, 1)$, labeled by us $G^{(\alpha)}$ ($\alpha = 0, 1, 2, 3, 4$), which have the special property that any momentum-dependent transformation (1.13) within a given $G^{(\alpha)}$ leaves the equation linear in the corresponding $p_\alpha$ (where we write $p_4 = m$).

As might be expected, the five classes of canonical transformations presented in BC fall into these five subgroups and, in fact, every transformation in a given $G^{(\alpha)}$ also preserves the corresponding mode of the scalar product. This we show in Secs. 4, 5, and 6, where the subgroups $G^{(0)}$, $G^{(4)}$, and $G^{(3)}$ (as typical of $G^{(i)}$, $i = 1, 2, 3$) and, correspondingly, "$p_0$-, "$m$-, and "$p_3$-linear" forms of the equation, are discussed in more detail. [It is not shown that an arbitrary momentum-dependent $SO(4, 1)$ transformation is canonical—in fact it is not possible to write the scalar product (1.8a)–(1.8c) in $SO(4, 1)$-invariant form.]

We find that $G^{(0)}$ is the maximal compact subgroup $SO(4)$, and $G^{(4)}$ the $SO(3, 1)$ group relating to the Lorentz transformation properties of the equation. The $G^{(i)}$ are also $SO(3, 1)$ subgroups, distinct from $G^{(4)}$ and from one another.

In Sec. 7, a discussion is given of the "little group" of $SO(4, 1)$ transformations which leave a particular transform of the equation invariant, and the nature of ambiguities which arise when one wishes to transform one equation into another are made explicit.

## 2. DIRAC EQUATION AND $SO(4, 1)$

The sixteen elements

$$I, \gamma_\mu, \gamma_5, \gamma_5\gamma_\mu, \quad \text{and} \quad [\gamma_\mu, \gamma_\nu] \quad (2.1)$$

of the Dirac–Clifford algebra form a complete set of $4 \times 4$ matrices, in terms of which the infinitesimal generators of a four-dimensional representation of any Lie group can be expressed as linear combinations with complex coefficients.

In this connection, one is familiar with the case of $SO(3, 1)$,[13] where the generators are defined as

$$S_{\mu\nu} = (i/4)[\gamma_\mu, \gamma_\nu] \quad (2.2)$$

and satisfy the characteristic Lorentz-group commutation rules

$$[S_{\mu\nu}, S_{\rho\sigma}] = -i(g_{\mu\rho}S_{\nu\sigma} + g_{\nu\sigma}S_{\mu\rho} - g_{\mu\sigma}S_{\nu\rho} - g_{\nu\rho}S_{\mu\sigma}). \quad (2.3)$$

The significance of these operators in regard to the Dirac equation is well known. In essence, the invariance of the Dirac description of free spin-$\frac{1}{2}$ particles under restricted homogeneous Lorentz transformations is expressed in the fact that

$$[\gamma_\mu p^\mu, J_{\rho\sigma}] = 0, \quad (2.4)$$

where

$$J_{\mu\nu} = L_{\mu\nu} + S_{\mu\nu} \quad (2.5)$$

and the $L_{\mu\nu}$, satisfying commutation rules analogous to (2.3), are defined by

$$L_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu. \quad (2.6)$$

Although not always referred to as such explicitly, representations of the Lie algebras of larger groups, such as $SO(4, 1)$ and $SO(4, 2)$, have been given in terms of (2.1) by, for example, Eddington,[1] Dirac,[2]

[13] We have taken some license with notation in referring to the groups $SO(3, 1)$, $SO(4)$, and $SO(4, 1)$, when in fact the covering groups $SL(2, C)$, $SU(2) \otimes SU(2)$, etc., are meant.

Harish-Chandra,[3] and Barut.[4] The connection between the orthogonal groups in five dimensions [such as $SO(4,1)$] and a class of relativistic wave equations, of which the Dirac equation is the simplest, was first discussed in detail by Lubański.[5] More recently $S\tilde{U}(4)$ [of which $SO(4,2)$ may be regarded as a type] has received attention from several authors[6]; and Barut,[7] in particular, has exploited the connection between Eq. (1.1) and $SO(4,2)$ in "reformulating the Dirac theory of the electron."

For our purposes here it will be sufficient to indicate and use the relationship of the group $SO(4,1)$ to the Dirac equation. Multiplying (1.1) by $\gamma_5$, one obtains

$$(\gamma_5\gamma_\mu p^\mu - \gamma_5 m)\psi^{(D)} = 0. \tag{2.7}$$

Defining

$$\Gamma_\mu = \gamma_5\gamma_\mu, \quad \Gamma_4 = \gamma_5, \tag{2.8}$$

and

$$p^4 = -m, \tag{2.9}$$

one can write (2.7) as

$$\Gamma_\alpha p^\alpha \psi^{(D)} = 0, \tag{2.10}$$

where the summation is now over $\alpha = 0, 1, 2, 3$, and 4. In the following, note that indices take these values:

$$\begin{aligned} &\alpha, \beta, \gamma, \delta, \epsilon: 0, 1, 2, 3, 4, \\ &\mu, \nu, \rho, \sigma: 0, 1, 2, 3, \\ &\tau, \zeta, \eta: 0, 1, 2, 4, \\ &\mathrm{a, b, c}: 1, 2, 3, 4, \\ &i, j, k: 1, 2, 3. \end{aligned}$$

Introducing $g^{\alpha\beta}$ $(= g_{\alpha\beta})$, with

$$g_{00} = -g_{11} = -g_{22} = -g_{33} = -g_{44} = 1, \quad g_{\alpha\beta} = 0, \quad \alpha \neq \beta, \tag{2.11}$$

we define $\Gamma^\alpha = g^{\alpha\beta}\Gamma_\beta$, etc. Now defining also

$$T_{\alpha\beta} = (i/4)[\Gamma_\alpha, \Gamma_\beta] \tag{2.12}$$

and noting

$$\{\Gamma_\alpha, \Gamma_\beta\} = 2g_{\alpha\beta}, \tag{2.13}$$

we find

$$[T_{\alpha\beta}, T_{\gamma\delta}] = -i(g_{\alpha\gamma}T_{\beta\delta} + g_{\beta\delta}T_{\alpha\gamma} - g_{\alpha\delta}T_{\beta\gamma} - g_{\beta\gamma}T_{\alpha\delta}), \tag{2.14}$$

which are the characteristic commutation rules for the Lie algebra of $SO(4,1)$. Since this group is noncompact, the $T_{\alpha\beta}$ are not all Hermitian, but

$$\Gamma_0 T^\dagger_{\alpha\beta} = T_{\alpha\beta}\Gamma_0. \tag{2.15}$$

One also has

$$[\Gamma_\alpha, T_{\beta\gamma}] = i(g_{\alpha\beta}\Gamma_\gamma - g_{\alpha\gamma}\Gamma_\beta). \tag{2.16}$$

Note that the ten different $T_{\alpha\beta}$ consist of the six different $T_{\mu\nu}$ [$\equiv S_{\mu\nu}$ of (2.2)] and four $T_{4\mu}$ [$= -(i/2)\gamma_\mu$], so that this Lie algebra is as small as any containing scalar multiples of all four Dirac matrices $\gamma_\mu$.[4] The representation of $SO(4,1)$ generated by these ten operators is irreducible. The Casimir operators[14] $-\frac{1}{2}T_{\alpha\beta}T^{\alpha\beta}$ and $-\omega_\alpha\omega^\alpha$, where

$$\omega_\alpha = \tfrac{1}{8}\epsilon_{\alpha\beta\gamma\delta\epsilon}T^{\beta\gamma}T^{\delta\epsilon} \tag{2.17}$$

($= \frac{3}{4}\Gamma_\alpha$ in this case), are multiples of the unit matrix by $-\frac{5}{2}$ and $-\frac{45}{16}$, respectively.

It is worth mentioning that there are two inequivalent irreducible representations of the Clifford algebra defined by (2.13), both of four dimensions. By making the choice (2.8) for $\Gamma_\alpha$, we fix on one of these. The other representation is obtained if one chooses instead

$$\Gamma_\mu = \gamma_5\gamma_\mu, \quad \Gamma_4 = -\gamma_5 \tag{2.18}$$

(and so necessarily $p^4 = m$). The set of $T_{\alpha\beta}$ one obtains in this case is then also different, again with

$$T_{\mu\nu} = (i/4)[\gamma_\mu, \gamma_\nu], \tag{2.19}$$

but now

$$T_{4\mu} = +(i/2)\gamma_\mu. \tag{2.20}$$

However, these $T_{\alpha\beta}$ generate an *equivalent* representation of $SO(4,1)$. (The invariants take the same values.) This is clear from the fact that this second set of $T_{\alpha\beta}$ is obtained from the first set via the substitution $\gamma_\mu \to -\gamma_\mu$. However, this can be achieved by a similarity transformation, because, under this substitution, a different set of matrices satisfying (1.3) is obtained and, as is well known, all irreducible representations of the Dirac–Clifford algebra are equivalent.

### 3. $SO(4,1)$ TRANSFORMATIONS OF THE EQUATION

When written in the form (2.10), the Dirac equation has an $SO(4,1)$-invariant appearance. One might hope to find operators $M_{\beta\gamma}$ satisfying commutation rules analogous to (2.14), and such that $\tilde{p}$ transforms[15] as a five-vector operator with respect to transformations generated by them, that is, such that [cf. (2.16)]

$$[p_\alpha, M_{\beta\gamma}] = i(g_{\alpha\beta}p_\gamma - g_{\alpha\gamma}p_\beta). \tag{3.1}$$

One would then have, ensuring $SO(4,1)$ invariance,

$$[\Gamma_\alpha p^\alpha, K_{\beta\gamma}] = 0, \tag{3.2}$$

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[14] See, for example, T. D. Newton, Ann. Math. **51**, 730 (1950).

[15] We introduce at this point the notation $\tilde{p}$ for the object with components $p_\alpha$, distinguishing it from the four- and three-vector operators $p$ and **p**, respectively.

with

$$K_{\beta\gamma} = M_{\beta\gamma} + T_{\beta\gamma}; \tag{3.3}$$

however, such $M^{\beta\gamma}$ (and $K^{\beta\gamma}$) cannot be found, as is clear from (3.1) and the fact that $p_4$ is a constant. Nevertheless, one can consider the effect of transformations generated by all ten $T_{\beta\gamma}$ on the equation. Furthermore, despite the above conclusions, we shall see that it is in some ways convenient to regard $\tilde{p}$ as a five-vector quantity and, similarly,

$$p_\alpha p^\alpha \equiv (p_0)^2 - (p_1)^2 - (p_2)^2 - (p_3)^2 - m^2 \tag{3.4}$$

as an $SO(4, 1)$ scalar.

Consider the $SO(4, 1)$ transformation

$$\Gamma_\alpha \to \Gamma'_\alpha = L_\alpha^\beta \Gamma_\beta \, [= (L\Gamma)_\alpha], \tag{3.5}$$

where $L_\alpha^\beta$ are real and satisfy

$$L_\alpha^\beta g^{\alpha\gamma} L_\gamma^\delta = g^{\beta\delta}, \tag{3.6}$$

$$L_0^0 \geq 1, \tag{3.7}$$

and

$$\det L = 1. \tag{3.8}$$

Then one can write

$$\Gamma'_\alpha = Q\Gamma_\alpha Q^{-1}, \tag{3.9}$$

where

$$Q = \exp\,[(i/2)\omega^{\alpha\beta} T_{\alpha\beta}], \tag{3.10}$$

with the $\omega^{\alpha\beta}$ $(= -\omega^{\beta\alpha})$ real quantities determined by the $L_\gamma^\delta$. [The converse result also holds: (3.9) and (3.10) $\Rightarrow$ (3.5)–(3.8).] Defining

$$\psi'(x) = Q\psi^{(D)}(x), \tag{3.11}$$

one obtains from (2.10) and (3.9) the equation

$$(L\Gamma)_\alpha p^\alpha \psi' = 0 \tag{3.12}$$

or, equivalently [using (3.6)],

$$\Gamma_\alpha (L^{-1}\tilde{p})^\alpha \psi' = 0. \tag{3.12$'$}$$

Thus this transformed equation is obtained from (2.10) by replacing therein the "five-vector" $\tilde{p}$ with its transform under the $SO(4, 1)$ transformation inverse to the $L$ of (3.5) [at the same time replacing $\psi^{(D)}$ by $\psi'$ as in (3.11)]. Conversely, if $\tilde{p}'$ is obtained from $\tilde{p}$ by some arbitrary $SO(4, 1)$ transformation, then the equation

$$\Gamma_\alpha p'^\alpha \psi' = 0 \tag{3.13}$$

can be obtained from (2.10) by defining $\psi'$ as in (3.11) with appropriate coefficients $\omega^{\alpha\beta}$ determining $Q$. Furthermore, since the $p_\alpha$ behave like real numbers to the extent that they commute with one another and with all $\Gamma_\beta$, and have only real eigenvalues on the functions under consideration, a generalization to allow $\omega^{\alpha\beta}$ to be (Hermitian) functions of $\tilde{p}$ is possible.

*To summarize:* The possible forms (3.13) of Dirac's equation, obtained from (2.10) via transformations of the form (3.11), with $\omega^{\alpha\beta} = \omega^{\alpha\beta}(\tilde{p})$, are determined by the possible transforms $\tilde{p}'$ of $\tilde{p}$,

$$p'_\alpha = L_\alpha^\beta p_\beta, \tag{3.14}$$

where $L_\alpha^\beta$ are (Hermitian) functions of $\tilde{p}$ satisfying (3.6–3.8). For all such transformations,

$$p'_\alpha p'^\alpha = p_\alpha p^\alpha \equiv p_\mu p^\mu - m^2, \tag{3.15}$$

and

$$\epsilon(p'_0) = \epsilon(p_0). \tag{3.16}$$

At this point we note that knowledge of $\tilde{p}'$ is not sufficient to uniquely determine the transformation $L_\alpha^\beta$ of (3.14), as for any $\tilde{p}'$ there is a "little group" of such transformations which leave it invariant. Correspondingly, there are transformations of the form (3.10), which, on application to a solution of a given equation (3.13), produce a further solution of the same equation. A further discussion of these questions is given in Sec. 7.

As mentioned in the Introduction, there are five subgroups, which we label $G^{(\alpha)}$, of $SO(4, 1)$, having particular significance when the question is raised of the canonicality of transformations of the form (3.11). $G^{(\alpha)}$ is that subgroup consisting of all $SO(4, 1)$ transformations which leave invariant arbitrary five-vectors whose only nonzero component is the $\alpha$th. Since every component of the "five-vector" $\tilde{p}$ appears linearly in (2.10), it follows that if $\tilde{p}'$ is obtained from $\tilde{p}$ by an $SO(4, 1)$ transformation (3.14) in $G^{(\beta)}$, the $\beta$ component of $\tilde{p}$ appears linearly also in (3.13), which we then refer to as "a '$p_\beta$-linear' form of the Dirac equation."

## 4. $G^{(0)}$ AND "$p_0$-LINEAR" FORMS

The subgroup $G^{(0)}$ [the maximal compact subgroup $SO(4)$] acts only on the indices 1, 2, 3, and 4. The corresponding generators are $T_{ab}$, which are Hermitian matrices, and they in fact generate two inequivalent unitary irreducible representations of $SO(4)$, labeled by the two eigenvalues $\pm 1$ of $\Gamma_0$ $(= \gamma_5\gamma_0)$, which is effectively a Casimir operator for this subgroup. (Note $[\Gamma_0, T_{ab}] = 0$.) Under the associated transformations (3.14), $p_0$ and $p_a p^a$ $[= -\omega^2(\mathbf{p}, m)]$ remain separately invariant.

Thus, from

$$\tilde{p} = (p_0, p_1, p_2, p_3, m) \tag{4.1}$$

via $G^{(0)}$ transformations, one can obtain

$$\tilde{p}' = (p_0, p'_1, p'_2, p'_3, p'_4), \tag{4.2}$$

where the $p'_a(p_b)$ are Hermitian and

$$p'_a p'^a = -\omega^2(\mathbf{p}, m). \tag{4.3}$$

The corresponding equation (3.13) is in each case

$$(\Gamma_0 p^0 + \Gamma_a p'^a)\psi' = 0, \tag{4.4}$$

which gives, on multiplication with $\Gamma_0$, the general "$p_0$-linear" form

$$p_0\psi' = \gamma_0(\gamma_1 p'_1 + \gamma_2 p'_2 + \gamma_3 p'_3 + p'_4)\psi'. \tag{4.4'}$$

In considering the canonicality of momentum-dependent transformations in $G^{(0)}$, we note that when

$$\psi^{(D)}(x) \to Q(p, m)\psi^{(D)}(x), \tag{4.5}$$

where $Q(p, m)$ is as in (3.10) [with $\omega^{\alpha\beta} = \omega^{\alpha\beta}(\tilde{p})$], one has, from (1.6),

$$\chi^{(D)\pm}(\mathbf{k}) \to Q(k, m)\chi^{(D)\pm}(\mathbf{k}). \tag{4.6}$$

Furthermore, when in particular $Q$ is in $G^{(0)}$, it follows from the Hermiticity of $T_{ab}$ that $Q(k, m)$ is a unitary matrix. Every such transformation is therefore canonical, preserving the scalar product in the mode (1.8a).

All transformations presented in BC and yielding "$p_0$-linear" forms are of $G^{(0)}$ type. For example, in the simple cases of the Foldy–Wouthuysen–Tani equation, which corresponds to

$$\tilde{p}' = (p_0, 0, 0, 0, \omega(\mathbf{p}, m)), \tag{4.7}$$

and the Mendlowitz equation, which corresponds to

$$\tilde{p} = \left(p_0, \frac{\omega(\mathbf{p}, m)}{|\mathbf{p}|} p_1, \frac{\omega(\mathbf{p}, m)}{|\mathbf{p}|} p_2, \frac{\omega(\mathbf{p}, m)}{|\mathbf{p}|} p_3, 0\right), \tag{4.8}$$

one sees that the corresponding transformations (1.17), (1.20) are indeed of the form

$$\exp\,[(i/2)\omega^{ab}T_{ab}], \tag{4.9}$$

with the $\omega^{ab}$ [$= \omega^{ab}(p_c)$] Hermitian. They are, in fact, the $SO(4, 1)$ transformations (3.10) corresponding to the "little group rotation-free" (l.g.r.f.) transformations $\tilde{p} \to \tilde{p}'$ in the two cases. More generally, corresponding to

$$\tilde{p}' = (p_0, \pm rq_1, \pm rq_2, \pm rq_3, \pm rq_4), \tag{4.10}$$

where the $q_a(p_b)$ are Hermitian,

$$r = \omega(\mathbf{p}, m)[-q_a q^a]^{-\frac{1}{2}}, \tag{4.11}$$

and

$$q_a(p^a - q^a) = 0, \tag{4.12}$$

from (4.4′) one has

$$p_0\psi' = \pm\omega(\mathbf{p}, m)[-q_a q^a]^{-\frac{1}{2}}\gamma_0(\gamma_1 q_1 + \gamma_2 q_2 + \gamma_3 q_3 + q_4)\psi', \tag{4.13}$$

which is the general "$p_0$-linear" form obtained in BC. [There are the $\tilde{p}'$ as in (4.2) which cannot be expressed in the manner of (4.10)–(4.12), viz., those for which $p'_a p^a = 0$. The corresponding transformations and equations were not obtained in BC.] Again, the transformation presented in BC and yielding (4.13) corresponds to the l.g.r.f. transformation of $\tilde{p}$ into $\tilde{p}'$ as in (4.10).

The angles appearing in (1.17) and (1.20) can be regarded as those between the Euclidean "four vectors" $p_a$ and $p'_a$ through which one rotates to obtain $p'_a$ in each case. [The idea of looking upon the Foldy–Wouthuysen–Tani and Mendlowitz transformations as rotations is not new,[16] nor is the use of the group $G^{(0)}$ in discussing them: it is evident in the work of Bollini and Giambiagi,[17] who have not, however, noted the connection with $SO(4, 1)$.]

### 5. $G^{(4)}$ AND "$m$-LINEAR" FORMS

$G^{(4)}$ is the $SO(3, 1)$ group associated with homogeneous Lorentz transformations. Thus momentum-dependent $SO(4, 1)$ transformations (3.14) in $G^{(4)}$ leave $p_4$ ($= m$), $p_\mu p^\mu$, and also $\epsilon(p_0)$ separately invariant. The associated generators $T_{\mu\nu}$ are not all Hermitian, but satisfy

$$T^\dagger_{\mu\nu} = \Gamma_0\Gamma_4 T_{\mu\nu}\Gamma_0\Gamma_4\, (= \gamma_0 T_{\mu\nu}\gamma_0) \tag{5.1}$$

and, as is well known, they generate two inequivalent irreducible representations of $SO(3, 1)$, labeled by the two eigenvalues $\pm i$ of $\Gamma_4$ ($= \gamma_5$) (cf. the case of $G^{(0)}$).

From

$$\tilde{p} = (p_0, p_1, p_2, p_3, m), \tag{5.2}$$

via $G^{(4)}$ transformations, one can obtain

$$\tilde{p}' = (p'_0, p'_1, p'_2, p'_3, m), \tag{5.3}$$

where the $p'_\mu(p_\nu)$ are Hermitian,

$$p'_\mu p'^\mu = p_\nu p^\nu, \tag{5.4}$$

and

$$\epsilon(p'_0) = \epsilon(p_0). \tag{5.5}$$

In each case, Eq. (3.13) is

$$(\Gamma_\mu p'^\mu - \Gamma_4 m)\psi' = 0, \tag{5.6}$$

yielding the general "$m$-linear" form

$$m\psi' = \gamma_\mu p'^\mu \psi'. \tag{5.6'}$$

It should be mentioned at this point that, because of the way it is obtained, $p'_\mu$ will not, in general,

[16] See, for example, K. M. Case, Phys. Rev. **95**, 1323 (1954).

[17] C. G. Bollini and J. J. Giambiagi, Nuovo Cimento **21**, 107 (1961). See also Ref. 19. *Note added in proof:* Since the preparation of this paper, E. de Vries [Physica **43**, 45 (1969)] has independently established this connection.

be a Lorentz four-vector operator like $p_\mu$. Our notation is perhaps misleading in this regard: $\tilde{p}$ is better regarded as a numerical five-vector than a five-vector operator for the purposes of this paper.

Each momentum-dependent transformation in $G^{(4)}$ is also canonical. This follows from the fact that (5.1) implies that the corresponding $Q(k, m)$ of (4.6) satisfies

$$Q^{-1}(k, m) = \gamma_0 Q^\dagger(k, m)\gamma_0, \tag{5.7}$$

so that the scalar product in the mode (1.8b) is preserved in every case.

Every transformation presented in BC and leading to an "$m$-linear" equation is of $G^{(4)}$ type (again corresponding to the l.g.r.f. transformation of $\tilde{p}$ into $\tilde{p}'$ in each case). Thus, for example, in the Chakrabarti case, where

$$\tilde{p}' = (\epsilon(p_0)(p_\mu p^\mu)^{\frac{1}{2}}, 0, 0, 0, m), \tag{5.8}$$

the l.g.r.f. transformation (1.24) is indeed of the form

$$\exp\{(i/2)\omega^{\mu\nu}T_{\mu\nu}\}. \tag{5.9}$$

In fact, *all* $\tilde{p}'$ of the form (5.3) can be written as

$$\tilde{p}' = (rq_0, rq_1, rq_2, rq_3, m), \tag{5.10}$$

with $q_\mu(p_\nu)$ Hermitian, $q_\mu q^\mu$ positive-definite,

$$r = (p_\mu p^\mu)^{\frac{1}{2}}(q_\nu q^\nu)^{-\frac{1}{2}}, \tag{5.11}$$

and

$$q_\mu(p^\mu - q^\mu) = 0. \tag{5.12}$$

[*Proof:* Take $q_\rho = p'_\mu p^\mu(p_\nu p^\nu)^{-1}p'_\rho$, noting that $p'_\mu p^\mu$ is positive-definite because of (5.5).] Then (5.6′) becomes

$$m\psi' = (p_\mu p^\mu)^{\frac{1}{2}}(q_\nu q^\nu)^{-\frac{1}{2}}\gamma_\rho q^\rho\psi', \tag{5.13}$$

which is the general "$m$-linear" form presented in BC.

Whereas in the case of the compact subgroup $G^{(0)}$ one can talk of an "angle of rotation" for each transformation, here one typically has pseudoangles associated with the anti-Hermitian $T_{0i}$, generators of "boosts" rather than rotations [cf. (1.24)].

## 6. $G^{(3)}$ AND "$p_3$-LINEAR" FORMS

Each of the three subgroups $G^{(i)}$ is again an $SO(3, 1)$ group. Transformations of the form (3.14) within $G^{(3)}$ leave $p_3$, $p_\tau p^\tau$, and $\epsilon(p_0)$ separately invariant. The corresponding generators $T_{\tau\eta}$ in this case are not all Hermitian, but

$$T^\dagger_{\tau\eta} = \Gamma_0\Gamma_3 T_{\tau\eta}\Gamma_0\Gamma_3 \,(= \gamma_0\gamma_3 T_{\tau\eta}\gamma_0\gamma_3). \tag{6.1}$$

In complete analogy with the case of $G^{(4)}$, the $T_{\tau\eta}$ generate two inequivalent irreducible representations of $SO(3, 1)$, labeled by the two eigenvalues $\pm i$ of $\Gamma_3\,(= \gamma_5\gamma_3)$ in this case.

From

$$\tilde{p} = (p_0, p_1, p_2, p_3, m), \tag{6.2}$$

via $G^{(3)}$ transformations, one can obtain

$$\tilde{p}' = (p'_0, p'_1, p'_2, p_3, p'_4), \tag{6.3}$$

where the $p'_\tau(p_\eta)$ are Hermitian,

$$p'_\tau p'^\tau = p_\eta p^\eta \,[= \lambda^2(p_0, p_1, p_2, m)], \tag{6.4}$$

and

$$\epsilon(p'_0) = \epsilon(p_0). \tag{6.5}$$

(Note that the transformations may become singular as $p_\tau p^\tau \to 0$.) The corresponding equation (3.13) in each case is

$$(\Gamma_\tau p'^\tau - \Gamma_3 p_3)\psi' = 0 \tag{6.6}$$

or, equivalently, the general "$p_3$-linear" form

$$p_3\psi' = -\gamma_3(\gamma_0 p'_0 - \gamma_1 p'_1 - \gamma_2 p'_2 - p'_4)\psi'. \tag{6.6′}$$

Again in analogy with the $G^{(4)}$ case, one finds that every momentum-dependent transformation in $G^{(3)}$ is canonical, the scalar-product mode [(1.8c); $i = 3$] being preserved in each case as a result of (6.1).

The transformations presented in BC and leading to "$p_3$-linear" equations are all of $G^{(3)}$ type (in each case corresponding to the l.g.r.f. transformation of $\tilde{p}$ into $\tilde{p}'$). In the simplest case, for example, where the equation is (1.25), corresponding to

$$\tilde{p}' = (\epsilon(p_0)(p_\tau p^\tau)^{\frac{1}{2}}, 0, 0, p_3, 0), \tag{6.7}$$

the l.g.r.f. transformation (1.28) is indeed of the form

$$\exp\{(i/2)\omega^{\tau\eta}T_{\tau\eta}\}. \tag{6.8}$$

Furthermore, all $\tilde{p}'$ as in (6.3) can be written in the form

$$\tilde{p}' = (rq_0, rq_1, rq_2, p_3, rq_4), \tag{6.9}$$

where $q_\tau(p_\eta)$ are Hermitian, $q_\tau q^\tau$ is positive-definite,

$$r = (p_\tau p^\tau)^{\frac{1}{2}}(q_\eta q^\eta)^{-\frac{1}{2}}, \tag{6.10}$$

and

$$q_\tau(p^\tau - q^\tau) = 0. \tag{6.11}$$

[Take $q_\zeta = p'_\tau p^\tau(p_\eta p^\eta)^{-1}p'_\zeta$.] Equation (6.6′) then becomes the general "$p_3$-linear" form of BC:

$$p_3\psi' = -\lambda(p_0, p_1, p_2, m)(q_\tau q^\tau)^{-\frac{1}{2}} \times \gamma_3(\gamma_0 q_0 - \gamma_1 q_1 - \gamma_2 q_2 - q_4)\psi'. \tag{6.12}$$

Again pseudoangles rather than angles appear in association with the anti-Hermitian generators $T_{0\tau}$ [cf. (1.28)].

## 7. SIGNIFICANCE OF THE LITTLE GROUP

We have mentioned that for a given "five vector" $\tilde{p}'$ there is a little group of $SO(4,1)$ transformations which leave it invariant. We are dealing here only with functions on which $p'_\alpha p'^\alpha$ $(= p_\alpha p^\alpha = p_\mu p^\mu - m^2)$ vanishes. In order to identify the little group appropriate in this situation, consider the particular case (corresponding to the Foldy–Wouthuysen–Tani equation)

$$\tilde{p}' = (p_0, 0, 0, 0, \omega(\mathbf{p}, m)). \tag{7.1}$$

It is seen that the matrix generators corresponding to the little group in this case are

$$T_{12},\ T_{23},\quad \text{and}\quad T_{31}, \tag{7.2}$$

together with

$$T_{10} - \epsilon(p_0)T_{14},\quad T_{20} - \epsilon(p_0)T_{24},\quad \text{and}\quad T_{30} - \epsilon(p_0)T_{34}. \tag{7.3}$$

The Lie algebra of (7.2) and (7.3) is isomorphic to that of the three-dimensional Euclidean group, (7.2) being the generators of "rotations," and (7.3) of "translations."

However, from (3.13) and (7.1), we find that $\Gamma_0 - \epsilon(p_0)\Gamma_4$ vanishes on the wavefunctions involved here. Multiplying this by $(i/2)\Gamma_j$ $(j = 1, 2, \text{or } 3)$, we obtain the result that this is also true of each of the generators (7.3). Thus the little group is effectively reduced to $SU(2)$.[18]

We conclude that, for any given $\tilde{p}'$, the little group consists of an effective part, which is $SU(2)$, and an ineffective part. Any transformation in the ineffective part is unity when applied to a wavefunction satisfying (3.13), while one in the effective part produces a new function satisfying the same equation. Note that $\tilde{p}'$, as in (7.1), can be obtained from $\tilde{p}$ by a transformation in $G^{(0)}$ and that the effective little group generators in this case (7.2) also generate $G^{(0)}$ transformations. This indicates that if (3.13) is obtained from (2.10) by means of a canonical $SO(4,1)$ transformation (i.e., a transformation in one of the subgroups $G^{(\alpha)}$), a subsequent little-group transformation leaving (3.13) invariant is also canonical.

It is clear that any $\tilde{p}'$ and $\tilde{p}''$ obtained from $\tilde{p}$ by $SO(4,1)$ tranformations (3.14) must themselves be linked by a further such transformation. Furthermore, by a procedure analogous to that used in obtaining (3.13) from (2.10), it is possible to obtain the equation

$$\Gamma_\alpha p''^\alpha \psi'' = 0 \tag{7.4}$$

directly from

$$\Gamma_\alpha p'^\alpha \psi' = 0. \tag{7.5}$$

Denoting by $Q''$ and $Q'$ the operators (3.10) used to obtain $\psi''$ and $\psi'$, respectively, from $\psi^{(D)}$, we have, trivially,

$$\psi'' = Q''Q'^{-1}\psi'. \tag{7.6}$$

While the operator $Q''Q'^{-1}$ certainly corresponds to an $SO(4,1)$ transformation taking $\tilde{p}'$ into $\tilde{p}''$ and enables one to obtain (7.4) from (7.5), it will not in general correspond to the l.g.r.f. such transformation, even if $Q''$ and $Q'$ correspond to the l.g.r.f. transformations taking $\tilde{p}$ into $\tilde{p}''$, $\tilde{p}'$, respectively. More precisely, if we denote by $Q(\tilde{p}', \tilde{p})$, $Q(\tilde{p}'', \tilde{p})$, and $Q(\tilde{p}'', \tilde{p}')$ the operators (3.10) corresponding to the l.g.r.f. transformations taking $\tilde{p}$ into $\tilde{p}'$, $\tilde{p}$ into $\tilde{p}''$, and $\tilde{p}'$ into $\tilde{p}''$, respectively, then in general

$$Q(\tilde{p}'', \tilde{p}') = \Lambda Q(\tilde{p}'', \tilde{p})Q^{-1}(\tilde{p}', \tilde{p}), \tag{7.7}$$

where $\Lambda$ is also of the form (3.10) and corresponds to an $SO(4,1)$ transformation in the little group defined by $\tilde{p}''$. If $\Lambda$ is in the effective part of the little group, then $Q(\tilde{p}'', \tilde{p}')$ and $Q(\tilde{p}'', \tilde{p})Q^{-1}(\tilde{p}', \tilde{p})$ will differ on the wavefunctions $\psi'$; but if it is in the ineffective part, then these two operators, while perhaps differing formally, will produce the same result when applied to any such $\psi'$ satisfying (7.5).

As an example, consider the case when $\tilde{p}'$ is as in (7.1), and

$$\tilde{p}'' = (\epsilon(p_0)(p_\mu p^\mu)^{\frac{1}{2}}, 0, 0, 0, m), \tag{7.8}$$

corresponding to the Chakrabarti equation. It is seen that in this case the l.g.r.f. transformation taking $\tilde{p}'$ into $\tilde{p}''$ is in the 0–4 plane, and correspondingly,

$$Q(\tilde{p}'', \tilde{p}') = \exp(i\varphi T_{04}) \tag{7.9}$$

$$= \cosh(\tfrac{1}{2}\varphi) - \gamma_0 \sinh(\tfrac{1}{2}\varphi), \tag{7.9'}$$

where $\varphi(\tilde{p})$ is Hermitian. A straightforward calculation of the pseudoangle $\varphi$ involved in this "boost" transformation yields

$$\varphi(\tilde{p}) = \operatorname{arc\,tanh}\{\epsilon(p_0)[\mathcal{E}^2 - \mathcal{M}^2][\mathcal{E}^2 + \mathcal{M}^2]^{-1}\}, \tag{7.10}$$

where

$$\mathcal{E} = \tfrac{1}{2}[|p_0| + \omega(\mathbf{p}, m)] \tag{7.11}$$

and

$$\mathcal{M} = \tfrac{1}{2}[(p_\mu p^\mu)^{\frac{1}{2}} + m]. \tag{7.12}$$

Then

$$\cosh(\tfrac{1}{2}\varphi) = \tfrac{1}{2}\{[\mathcal{E}/\mathcal{M}]^{\frac{1}{2}} + [\mathcal{M}/\mathcal{E}]^{\frac{1}{2}}\} \tag{7.13}$$

and

$$\sinh(\tfrac{1}{2}\varphi) = [\tfrac{1}{2}\epsilon(p_0)]\{[\mathcal{E}/\mathcal{M}]^{\frac{1}{2}} - [\mathcal{M}/\mathcal{E}]^{\frac{1}{2}}\}. \tag{7.14}$$

[18] There is a marked analogy here with the case of the little group (in the usual connection with the Poincaré group now) appropriate to a particle of zero rest mass and nonzero spin, as treated by V. Bargmann and E. P. Wigner, Proc. Natl. Acad. Sci. U.S. **34**, 211 (1948). See also de Vos and Hilgevoord (Ref. 5) in this connection.

Because $\tilde{p}'$ in this case corresponds to the Foldy–Wouthuysen–Tani equation, Eq. (7.5) yields

$$\mathcal{M}\psi' = m\psi', \tag{7.15}$$

$$\mathcal{E}\psi' = \omega(\mathbf{p}, m)\psi', \tag{7.16}$$

and

$$\epsilon(p_0)\gamma_0\psi' = \psi'. \tag{7.17}$$

It follows then from (7.9′) and (7.13)–(7.17) that

$$Q(\tilde{p}'', \tilde{p}')\psi' = [m/\omega(\mathbf{p}, m)]^{\frac{1}{2}}\psi'. \tag{7.18}$$

However, in this example we have

$$Q(\tilde{p}', \tilde{p}) \equiv F(\mathbf{p}, m) \tag{7.19}$$

and

$$Q(\tilde{p}'', \tilde{p}) \equiv C(p), \tag{7.20}$$

and it is known that[19]

$$\psi^{(C)}(x)\,[= C(p)F^{-1}(\mathbf{p}, m)\psi^{(F)}(x)] = [m/\omega(\mathbf{p}, m)]^{\frac{1}{2}}\psi^{(F)}(x). \tag{7.21}$$

Thus from (7.18) we deduce

$$Q(\tilde{p}'', \tilde{p}')\psi^{(F)} = C(p)F^{-1}(\mathbf{p}, m)\psi^{(F)} \tag{7.22}$$

$$= Q(\tilde{p}'', \tilde{p})Q^{-1}(\tilde{p}', \tilde{p})\psi^{(F)}. \tag{7.22'}$$

On inspection, however, $Q(\tilde{p}'', \tilde{p}')$ and $Q(\tilde{p}'', \tilde{p}) \times Q^{-1}(\tilde{p}', \tilde{p})$ are found to be formally distinct, and we conclude that they are related in the manner (7.7), with $\Lambda$ in the ineffective part of the little group of $\tilde{p}''$.

## 8. CONCLUSION

The connection between the group $SO(4, 1)$ and the free-particle Dirac equation can be exploited to allow the presentation of a unified treatment of the well-known canonical transformations of the equation. Similarities and relationships between these transformations assume a new and simple significance in such a treatment.

This approach also makes obvious the existence and also the actual form of many other similar canonical transformations, some of which we feel will prove useful[8] in discussing limiting situations other than the nonrelativistic and extreme-relativistic ones (where the Foldy–Wouthuysen–Tani and the Mendlowitz transformations, respectively, are most appropriate).

It is tempting to speculate as to a deeper physical significance of the group $SO(4, 1)$ itself in this context, in view of recent activity centering on this and related groups in connection with dynamical symmetries[4,20] for elementary particles. However, there are relativistic wave equations, linear in the energy–momentum operators, for which there is no simple connection with $SO(4, 1)$.[21] For all such equations describing massive particles, it is, however, a consequence of Lorentz invariance that there will be a Chakrabarti-type transformation corresponding to the transformation of the four momentum to the rest frame: Such a transformation expresses the canonical Wigner amplitudes in terms of the manifestly covariant ones.

Foldy and Wouthuysen[10] generalized their approach to the free-particle Dirac equation to gain considerable insight into the problem of the Dirac particle in interaction with a weak electromagnetic field, and this approach has been pursued consequently by several authors.[22] The Foldy–Wouthuysen method involves a perturbation procedure and yields a "$p_0$-linear" equation containing an infinite number of terms. In the absence of the interaction, this equation reduces to their form of the free-particle equation. One of us (H. A. C.) has generalized this procedure to develop similar expansions corresponding to various other forms obtainable from the free-particle Dirac equation via canonical $SO(4, 1)$ transformations.

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[19] R. H. Good, Jr., and M. E. Rose, Nuovo Cimento **24**, 864 (1962).

[20] See, for example, A. Böhm, Phys. Rev. **175**, 1767 (1968); C. Fronsdal, Phys. Rev. **156**, 1665 (1967); Y. Nambu, Phys. Rev. **160**, 1171 (1967); A. O. Barut, Phys. Rev. Letters **26B**, 308 (1968).

[21] We are indebted to Professor C. A. Hurst for this information.

[22] See E. Eriksen and M. Kolsrud, Nuovo Cimento Suppl. **18**, 1 (1960), and references therein.