Harvey A. Cohen, *The Application of IFS [Iterated Function Systems] to Image Analysis*, Proceedings of the IEEE International Conference on Image Processing ICIP'89, Singapore, Sept 5-8, 1989, Vol 2, pp 583-587.

THE APPLICATION OF IFS [Iterated Function Systems] TO IMAGE ANALYSIS

Harvey A. Cohen

Computer Science Department, La Trobe University
Bundoora, Melbourne, Victoria, Australia 3095

The Iterative Function System [IFS] for encoding images has been recently developed by Barnsley, Demko and co-workers as a byte-efficient encoding of complex images. The utility of this encoding for purposes of image analysis is discussed. Transformation laws for IFS descriptions are developed, and ambiguities due to symmetry effects are described.

Introduction

Following their early success in the utilisation of IFS to describe fractals, Barnsley and Demko mounted an investigation of the applicability of the IFS scheme for the description of arbitrary shapes. Central to the (general) IFS has became what they refer to as the collage theorem: the notion that if a figure can be "lazy tiled" with sufficient accuracy by smaller copies of itself at various orientations *(contraction mappings)* then the figure can be generated to satisfactory accuracy by the IFS scheme. In a series of recent papers Barnsley, Demko and their co-workers have proposed a method for the description of arbitrary images through an encoding of image segments by means of sets of contraction mappings and associated probabilities, which they term the IFS (Iterated Functions System) description. From an IFS description, each segment can be reconstructed by a stochastic process by applying each mapping at its associated probability. This reconstruction process (image synthesis from IFS parameter description) lends itself to parallel implementation on processor networks, so that the decoding process is computationally tractable. By encoding manually real-world scenes, Barnsley, Demko, et al have shown that compressions from 2000 to 10,000 can be realised with acceptable accuracy. Barnsley and Demko have reported that work is proceeding on the development of an automated system for IFS encoding. With the promised availability of such highly compressed IFS images, the need to examine the practicality and difficulties of the use of IFS encoded images for computer vision and image processing becomes urgent. It is worth noting that one of the motivations that Barnsley and Demko offer for the development of the IFS encoding is that highly compressed images will be more suitable for computer vision.

In this paper problems and issues related to the use of IFS encoded images for 2-D recognition purposes are discussed

In the next section a general introduction to Iterated Function System [IFS] theory is presented. The following section is devoted to examples that demonstrate the non-uniqueness of IFS encoding. The following sections explore the problem of establishing that two IFS description do in fact describe the same segment at different orientations and location. We consider the comparision of two apparently different IFS parameter sets, with the same number of parameters. There are two questions vital to image analysis applications:
(a) Do the two IFS descriptions refer to the same object at the same location?
(b) Do the two IFS descriptions describe the same object at different locations (or congruent objects)?
The conclusion offers an overview of the future scope of IFS in image analysis.

Iterated Function System

Basic Theory of IFS

Informally one can define fractals as figures of which any fragment is similar to some fragment of any magnified or reduced scale version of itself. Deterministic fractals, such as the Hilbert curve and the snowflake curve had been known since the late nineteenth century: such deterministic fractals show precise self-similarity at all scales. Mandelbrot[1], in his provocatively entitled opus, *The Fractal Geometry of Nature* extended the notion of fractal to include stochastically modes of generation.

The classic means of generation of fractals curves is by mapping the motion of a point which transverses the entire curve. In contrast, Barnsley and Demko [1] presented a means for global description of fractals which they termed the Iterated Function System (IFS) scheme.

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The formal description of IFS is simplified by constraining attention to a system of unit dimensions, with 05x5..1, 0y5..1. Thus the usual pixel coordinates have to be replaced by the appropriate rational fractions, although extending particular formula to resealed coordinates is straightforward.

IFS descriptions utilise contraction mappings such as W given by:

$$W\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix}$$

where 0< 1 ad-bc I <1 . An Iterated Function System is a set of such transformations, each with an associated probability: [W_i,p_i | i=l..N,) where the sum of the N probabilities p_i is one. Note that an IFS description involving N mappings is specified by 7N parameters. An IFS set provides a compact description of what can be a highly complex image region.

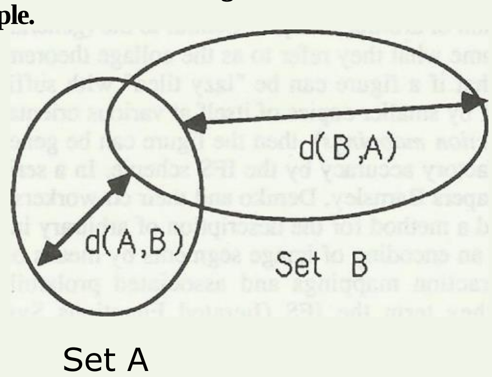
For later reference, we note that such a W has an associated *contractivity* equal to the absolute value of the largest eigenvalue of its matrix part. The entire IFS has *a contractivity index* equal to the maximum *contractivity* of the N contraction mappings W involved.

The process whereby an image region can be synthesised from an IFS parameter set is a stochastic iterative procedure, termed by Barnsley [6] 'random iteration'. The algorithm *is:*
From an arbitrary start point, traverse about the image by the repeated application of one of the contraction mappings: at reach iteration the choice of mapping utilised is purely random, with fixed probability. After the first fifty (or so) iterations, each pixel visited is marked; after some 1000 or so iterations, the set of marked pixels constitutes the attractor of the IFS.
The convergence of this random iteration process is independent of the starting point. Beyond a certain number of iterations, the marked regions does not alter.

For completeness it is noted that instead of applying the random iteration process described above to generate the attractor of a set of contraction mappings, the same attractor can be generated by a deterministic process somewhat akin to Conway's Game of Life. In the deterministic algorithm[6], a set of marked pixels constitutes one generation, and the deterministic algorithm determines how one generation is formed from the preceding generation. The algorithm specifies that each marked pixel marks those pixels in the next generation that it can be mapped into by ANY of the *N* transformations of the IFS. The next generation is thus determined by applying all the IFS transformations to all the marked pixels of the old generation. The process is then repeated until convergence (or approximate convergence) when the new generation is identical to the old. The availability of two such different algorithms is of much interest in the evolving theory of distributed and parallel computing; see Cohen [7]

Collage Theorem for IFS

Traditional fractal figures can be recognised as having components which are derived by contraction mappings on the entire figure. This feature of fractals, their self-similarity, does rather naturally lead to an IFS description if one seeks to give a global description of a fractal. However, Barnsley and co-workers [3][4][5][6] found that an IFS description of an arbitrary figure can be determined by means of what is termed the *Collage Theorem.* This theorem considers an operation of lazy tiling, where a figure is covered by contracted and rotated copies of itself, the covering being *lazy* in that the copies used for tiling may overlap, but the tiling needs to be as good as possible in covering and not overlapping the boundaries of the figure. the IFS scheme rests on the *Collage Theorem ::If a set of mappings lazy tiles a figure with N copies of itself, then the corresponding IFS system, with probabilities proportional to the contraction ratio utilised by each mapping is a satisfactory description of the figure.* The departure from an exact fit is given in terms of a Hausdorf metric. This metric specifies the distance between two sets A and B as the maximum, for points in set A, of the minimum distace between a point in A and any point in B. For simple closed figures the Hausdorf metric agrees with natural definitions of the distance between two figures. Consider the following example.



The Hausdorf distance between the sets A and B is computed as the maximum of d(A,B) and d(B,A), where d(A,B) is the maximum, for any point in set A, of the minimum distance to any point in set B. d(B,A) is similarly defined The Hausdorf distance is a direct measure of the (maximal) non-overlap of the two sets, and is a natural "difference" measure.

What has been proven (by Barnsley et al [6]) is that if a figure differs from a lazy tiled copy of itself by d, then an IFS representation will generate that figure with Hausdorf distance error of just d/(1-s) where s is the *contractivity index* of the IFS set.

So far in the above discussion it has been presumed that a given figure described by IFS has a uniform grey scale. However, if the figure is generated by the random algorithm, there is a flexibility in adjusting the probabilities so that the gray scale assigned to a figure is a

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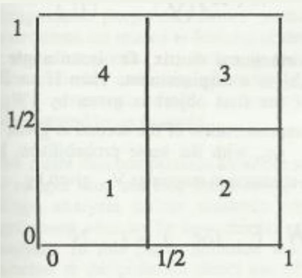
measure of the number of times a pixel has been visited. Of course, once gray scale is used as a measure of likelihood of pixel encounter, further "underlaping" collages may be added to the set to fully map the gray scale contouring.

Non-uniqueness of IFS

In assessing the potential application of IFS to computer vision and image processing, one needs to ask *Is an IFS encoding unique?* We demonstrate below that IFS encoding is not unique. The following question then arises: *Can a particular automated procedure for IFS yield a unique set of parameters.? We* cannot answer this question, as there simply is not an automated procedure for IFS encoding. However we point out that in recognition of 2-D shapes one is faced with the task of comparing figures of differing locations and orientation. A basic need is to develop means of making such transformations of IFS . To that end a new result showing how to verify the identity of two sets of IFS parameters of the same size is given.

The major aim of this section is to illustrate different ways in which an IFS description of a figure may be non-unique. To keep the discussion as simple as possible, the figure concerned is in fact the entire unit square, for which there is an IFS description, which provides also a natural (trivial) illustration of the collage theorem. A natural way of lazy tiling the unit square is to decompose it into four equal squares, with no overlap:



/2

The transformations that map the whole square into each of these four regions are W1 ,W2 ,W3, W4 , where

$$W_1\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}1/2 & 0\\0 & 1/2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}0\\0\end{pmatrix}$$

$$W_2\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}1/2 & 0\\0 & 1/2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}1/2\\0\end{pmatrix}$$

$$W_3\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}1/2 & 0\\0 & 1/2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}0\\1/2\end{pmatrix}$$

$$W_4\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}1/2 & 0\\0 & 1/2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}1/2\\1/2\end{pmatrix}$$

According to the Collage Theorem, the IFS parameter set is just { W_i,p_i | i=1..4,) where each probability p_i = 0.25. Each transformation involves six parameters, and coupled with each transformation is a probability, so that there is a 28 parameter in this IFS description of the unit square. Note that as the Hausdorf distance between the union of the four quadrants and the unit square is zero, the IFS description has zero error.

In the following sub-sections we show give various alternate IFS descriptions of the unit square which are based on non-overlapping collages.

Non-uniqueness due to Symmetry

This first example of non-uniqueness arises due to the rotational symmetry of a cube. During the lazy tiling, one may accordingly rotate the first cube through any multiple of 90 degrees before laying in the first quadrant. Thus in place of the W_i used above any of the following three contraction mappings may be used:

$$W_{1A}=\begin{pmatrix}0 & -1/2\\1/2 & 0\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}1/2\\0\end{pmatrix}$$

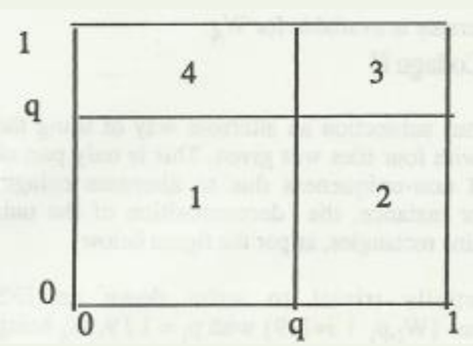
$$W_{1B}=\begin{pmatrix}-1/2 & 0\\0 & -1/2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}1/2\\1/2\end{pmatrix}$$

$$W_{1C}=\begin{pmatrix}0 & 1/2\\-1/2 & 0\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}0\\1/2\end{pmatrix}$$

The three mappings W_{1A} W_{1B} W_{1C} are in fact of the form of the transformation W1 followed by a rotation about the centroid of region 1. The measure of non-uniqueness revealed here is an immediate reflection of the rotational symmetry of the square.

Alternate Collage I

Consider the following partition of the unit square into four sub-regions 1,2,3 and 4.:



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In this alternate collage of the unit square, mappings for IFS scheme are the linear maps that map the unit square into the four sub-regions 1,2, 3, and 4. These mappings are W_1, W_2, W_3, W_4 as given by the formulas:

$$W_1\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}q & 0\\0 & q\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} + \begin{pmatrix}0\\0\end{pmatrix}$$

$$W_2\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}1-q & 0\\0 & q\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} + \begin{pmatrix}q\\0\end{pmatrix}$$

$$W_3\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}q & 0\\0 & 1-q\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} + \begin{pmatrix}0\\q\end{pmatrix}$$

$$W_4\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}1-q & 0\\0 & 1-q\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} + \begin{pmatrix}q\\q\end{pmatrix}$$

An IFS description of the unit square is thus (W_i, p_i | i=1..4) where each p_i is equal to the area of the corresponding rectangular segment, e.g. $p_1= q^2$.

Note that by taking into account the symmetry of the square segment 1, one might replace the W_i above by one of the following transformations:

$$W_{1A} = \begin{pmatrix}0 & -q\\q & 0\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} + \begin{pmatrix}q\\0\end{pmatrix}$$

$$W_{1B} = \begin{pmatrix}-q & 0\\0 & -q\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} + \begin{pmatrix}q\\q\end{pmatrix}$$

$$W_{1C} = \begin{pmatrix}0 & q\\-q & 0\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} + \begin{pmatrix}0\\q\end{pmatrix}$$

Similarly, alternate transforms are available replacing the W_3 above, as segment 3 is also square. Both segments 2 and 4 being rectangular, they are invariant under rotation by 180 degrees, as expressed in an alternate W_2:

$$W_{2A}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}0 & 1-q\\q & 0\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} + \begin{pmatrix}0\\q\end{pmatrix}$$

A similar alternate is available for W4.

Alternate Collage II

In the previous subsection an alternate way of tiling the unit square with four tiles was given. This is only part of the story of non-uniqueness due to alternate collage. Consider, for instance, the decomposition of the unit square into nine rectangles, as per the figure below.

It is essentially trivial to write down an IFS representation: (W_i,p_i | i=1..9) with p_i = 1 / 9, W_i being the contraction transform that maps the unit square into sub-square number i.

7	8	9
4	5	6
1	2	3

IFS Transformation Law

Suppose a particular IFS set is assigned to one image segment, and there is another segment, identical, but suffering translation and rotation, in the same (or other) image. How may an IFS parameter set for the second segment be derived from the first? That is, there are two identical instances at different orientations as well as being separated by a translation. The mapping from points in the first object to corresponding points in the second is given by an Euclidean transform:

$$E\begin{pmatrix}x\\y\end{pmatrix} = R(\Omega)\begin{pmatrix}x\\y\end{pmatrix} + \begin{pmatrix}a\\b\end{pmatrix}$$

where R is an orthogonal matrix, f is an angle of rotation, and (a,b) is a displacement. Then if an IFS representation of the first object is given by (W_i,p_i | i=1..N}, an IFS representation of the second is given by (V_i,p_i | i=1..N } i,e., with the same probabilities, but

with transformed contraction mappings V_i, given by

$$V_i = E^{-1}\, W_i\, E \quad \text{for } i = 1, N$$

where the inverse of E is defined as the transformation:

$$E^{-1}\begin{pmatrix}x\\y\end{pmatrix} = R(-\Omega)\begin{pmatrix}x-a\\y-b\end{pmatrix}$$

Identification Scheme

Consider an image object (segment) described by a particular IFS description: (W_i,p_i | i=1..N). In another image a potentially identical object is described by the IFS system, {V_i,q_i | i=1..N}. How is one to determine whether the same object is present in the two images? It is

assumed that in both IFS descriptions the same number of transformations are involved. The comparison involved is greatly simplified by the recognition that the determinant of the 2*2 matrices involved are invariant under the Euclidean transformation deduced above. Hence the first step in determining identity is determining whether the set of determinants of the two IFS systems are to some accuracy, identical. If the set of determinant values are non-degenerate, so that there is a unique mapping between one set and the next, then the identification task becomes the relatively straightforward task of determining the unique euclidean transform that can effect all the transformations.

Discussion

The remarkable data compression possible using the IFS has been demonstrated by the encoding using IFS of realistic scenes from the magazine *National Geographic:* [5],[6]. The actual encoding however was performed manually, after segmentation. These published examples clearly show gray scale and colour encoding via subtle graduations of colour on flowers and leaves for example. Data reduction even as high as 1000 to 1 have been reported in some examples.

In order to use an IFS description of an image segment(s) for recognition purposes one must have means for identifying with a template of some sort. In this paper a careful analysis of the capabilities of the IFS was performed. It was shown that IFS descriptions are ambiguous, and not immediately useful in general for recognition purposes. Some ambiguities in IFS descriptions are related to features of the Euclidean Group in two dimensions, as shown in the examples above. There is further ambiguity where spatial symmetries of an image segment admit alternate collages, as we demonstrated by an example.

The major data compression available using IFS encoding of images does not lead immediately to advantages for image analysis unless methods are developed for performing what are by now classsic operations of image analysis. In this paper attention has essentially been directed at the potential direct use of IFS encoding to identify image segments and objects. I have shown here the problems of non-uniqueness that arise in the simplest case where one is dealing with identically sized sets of IFS parameters.

Further problems of non-uniqueness arise where the IFS encoding has served to encode gray scale (or colour) as well as segment area. For example, the researcher may wish to somehow delineate a region of gray-scale range that lies within the regions described by two IFS parameter sets. How to do so without first synthesising the whole image and applying traditional techniques is not clear.

To apply IFS to general object recognition problems, will require the development of identification schemes linking IFS descriptions of objects viewed at different 3D perspective. This more general "identification" problem is clearly difficult, but some elaborated theory will be necessary if IFS description is to have other than a narrow role in image analysis.

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